

**Rourkela**



**PCEC4303–CONTROL SYSTEM ENGINEERING**

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1. Introduction to Control Systems: Basic elements of control system
  - Open loop and closed loop systems
  - Tracking System, Regulators
  - Differential equation
  - Transfer function
2. Modeling of electric systems: Translational and rotational mechanical systems
  - Block diagram reduction techniques.
  - Signal flow graph, Mason's Gain Formula.
3. Feedback characteristics of Control Systems: Effect of negative feedback on sensitivity.
  - Bandwidth, Disturbance.
  - Linearizing effect of feedback, Regenerative feedback.
4. Control Components: D.C. Servomotors, A.C. Servomotors.
  - A.C. Tachometer.
  - Synchros, Stepper Motors.

### Module-II

5. Time response Analysis: Standard Test Signals
  - Time response of first order systems to unit step and unit ramp inputs.
  - Time Response of Second order systems to unit step input.
  - Time Response specifications, Steady State Errors.
  - Generalised error series and Generalised error coefficients.
6. Stability Theory: Stability and Algebraic Criteria
  - Concept of stability, Necessary conditions of stability.
  - Hurwitz stability criterion, Routh stability criterion.
  - Application of the Routh stability criterion to linear feedback system.
  - Relative stability by shifting the origin in s-plane.
7. Root locus Technique: Root locus concepts, Rules of Construction of Root locus
  - Determination of Roots from Root locus for a specified open loop gain.
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8. Compensation Technique: Systems with transportation lag.
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9. Frequency Response Analysis: Frequency domain specifications
  - Correlation between Time and Frequency Response with

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- Constant M-circles.
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- Zeigler-Nichols method of tuning PID controllers.

## Module -1

Introduction to Control Systems: Basic elements of control system Open loop and closed loop systems, Tracking System, Regulators, and Differential equation, Transfer function. Modeling of electric systems - Translational and rotational mechanical systems. Block diagram reduction techniques. Signal flow graph, Mason's Gain Formula. Feedback characteristics of Control Systems: Effect of negative feedback on sensitivity. Bandwidth, Disturbance. Linearizing effect of feedback, Regenerative feedback. Control Components: D.C. Servomotors, A.C. Servomotors. A.C. Tachometer, Synchros, Stepper Motors.

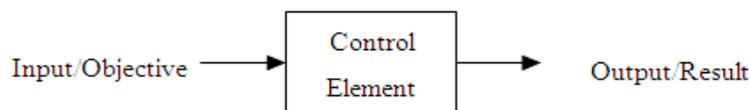
### Lecture-1

Basic Concepts of Control Systems, Open loop and closed loop systems

#### 1. Basic elements of control system:-

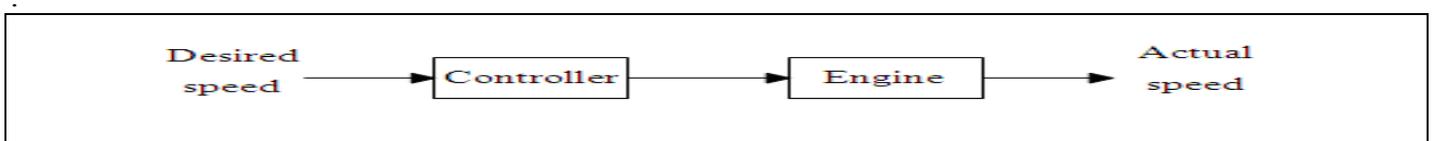
In recent years, control systems have gained an increasingly importance in the development and advancement of the modern civilization and technology. Figure shows the basic components of a control system. Disregard the complexity of the system; it consists of an input (objective), the control system and its output (result). Practically day-to-day activities are affected by some type of control systems. There are two main branches of control systems:

- 1) Open-loop systems and
- 2) Closed-loop systems.



#### Open-loop systems:

The open-loop system is also called the non-feedback system. This is the simpler of the two systems. A simple example is illustrated by the speed control of an automobile as shown in Figure 1-2. In this open-loop system, there is no way to ensure the actual speed is close to the desired speed automatically. The actual speed might be way off the desired speed because of the wind speed and/or road conditions, such as uphill or downhill etc. Example-Automatic washing Machine, immersion rod, A field control d.c motor and automatic control of traffic lamp.



(Fig1.2 Basic open-loop system)

#### Closed-loop systems:

The closed-loop system is also called the feedback system. A simple closed-system is shown in Figure 1-3. It has a mechanism to ensure the actual speed is close to the desired speed automatically. In closed loop control systems the control action is dependent on desired output .If any system having one or more feedback paths forming a closed loop system. Example-air conditioners are provided with thermostat.

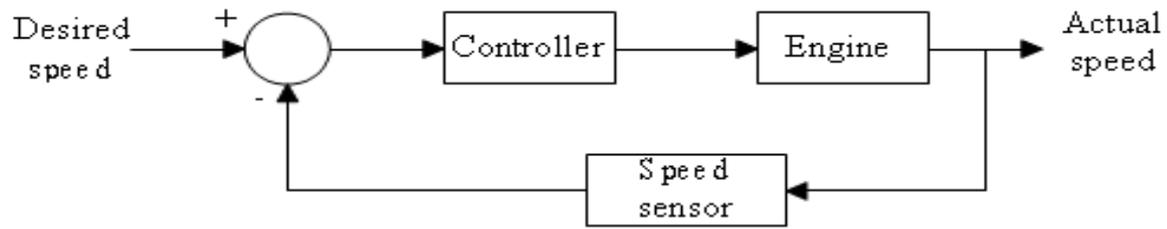


Fig. 1-3. Basic closed-loop system.

#### Transfer Function

A simpler system or element may be governed by first order or second order differential equation. When several elements are connected in sequence, say "n" elements, each one with first order, the total order of the system will be nth order. In general, a collection of components or system shall be represented by nth order differential equation

$$\frac{d^n y(t)}{dt^n} + a_{n-1} \frac{d^{n-1} y(t)}{dt^{n-1}} + \dots + a_0 y(t) = b_m \frac{d^m u(t)}{dt^m} + \dots + b_0 u(t)$$

In control systems, transfer function characterizes the input output relationship of components or systems that can be described by Linear Time Invariant Differential Equation.

In the earlier period, the input output relationship of a device was represented graphically. In a system having two or more components in sequence, it is very difficult to find graphical relation between the input of the first element and the output of the last element. This problem is solved by transfer function.

#### Definition of Transfer Function

Transfer function of a LTIV system is defined as the ratio of the Laplace Transform of the output variable to the Laplace Transform of the input variable assuming all the initial condition as zero.

#### Properties of Transfer Function

- The transfer function of a system is the mathematical model expressing the differential equation that relates the output to input of the system.
- The transfer function is the property of a system independent of magnitude and the nature of the input.
- The transfer function includes the transfer functions of the individual elements. But at the same time, it does not provide any information regarding physical structure of the system.
- The transfer functions of many physically different systems shall be identical.
- If the transfer function of the system is known, the output response can be studied for various types of inputs to understand the nature of the system.
- If the transfer function is unknown, it may be found out experimentally by applying known inputs to the device and studying the output of the system.

#### How you can obtain the transfer function (T. F.)

- Write the differential equation of the system.
- Take the L. T. of the differential equation, assuming all initial condition to be zero.
- Take the ratio of the output to the input. This ratio is the T. F.

#### Mathematical Model of control systems

A control system is a collection of physical object connected together to serve an objective. The mathematical model of a control system constitutes a set of differential equation.

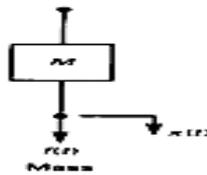
## 2. Modeling of electric systems:-

### Mechanical Translational systems

The model of mechanical translational systems can obtain by using three basic elements mass, spring and dashpot. When a force is applied to a translational mechanical system, it is opposed by opposing forces due to mass, friction and elasticity of the system. The force acting on a mechanical body is governed by Newton's second law of motion. For translational systems it states that the sum of forces acting on a body is zero.

### Force balance equations of idealized elements

Consider an ideal mass element shown in fig. which has negligible friction and elasticity. Let a force be applied on it. The mass will offer an opposing force which is proportional to acceleration of a body.



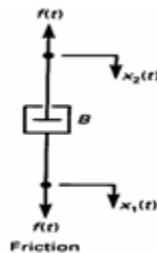
Let  $F$  = applied force

$F_m$  = opposing force due to mass

Here  $F_m \propto \frac{d^2x}{dt^2}$

By Newton's second law,  $F = F_m = M \frac{d^2x}{dt^2}$

Consider an ideal frictional element dash-pot shown in fig. which has negligible mass and elasticity. Let a force be applied on it. The dashpot will be offer an opposing force which is proportional to velocity of the body.



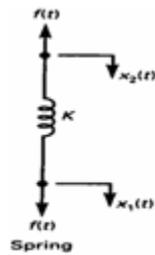
Let  $F$  = applied force

$F_b$  = opposing force due to friction

Here,  $F_b \propto \frac{dx}{dt}$

By Newton's second law,  $F = F_b = B \frac{dx}{dt}$

Consider an ideal elastic element spring shown in fig. which has negligible mass and friction.



Let  $F$  = applied force

$F_k$  = opposing force due to elasticity

Here,  $F_k \propto x$

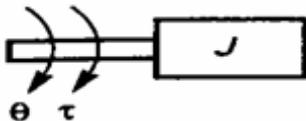
By Newton's second law,  $F = F_k = Kx$

### Mechanical Rotational Systems

The model of rotational mechanical systems can be obtained by using three elements, moment of inertia [J] of mass, dash pot with rotational frictional coefficient [B] and torsion spring with stiffness[k]. When a torque is applied to a rotational mechanical system, and it is opposed by opposing torques due to moment of inertia, friction and elasticity of the system. The torque acting on rotational mechanical bodies is governed by Newton's second law of motion for rotational systems.

### Torque balance equations of idealized elements

Consider an ideal mass element shown in fig. which has negligible friction and elasticity. The opposing torque due to moment of inertia is proportional to the angular acceleration.



Let  $T$  = applied torque

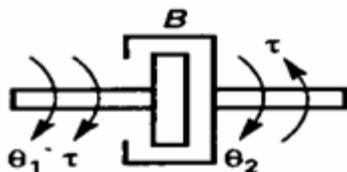
$T_j$  = opposing torque due to moment of inertia of the body

Here,  $T_j \propto d^2 \theta / dt^2$

By Newton's law

$$T = T_j = J d^2 \theta / dt^2$$

Consider an ideal frictional element dash pot shown in fig. which has negligible moment of inertia and elasticity. Let a torque be applied on it. The dash pot will offer an opposing torque is proportional to angular velocity of the body.



Let  $T$  = applied torque

$T_b$  = opposing torque due to friction

Here

$$T_b \propto \frac{d(\theta_1 - \theta_2)}{dt}$$

By Newton's law

$$T = T_k = B \frac{d(\theta_1 - \theta_2)}{dt}$$

Consider an ideal elastic element, torsion spring as shown in fig. which has negligible moment of inertia and friction. Let a torque be applied on it. The torsion spring will offer an opposing torque which is proportional to angular displacement of the body.



Let  $T$  = applied torque

$T_k$  = opposing torque due to friction

Here,

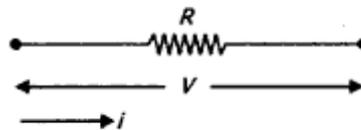
$$T_k \propto (\theta_1 - \theta_2)$$

By Newton's law

$$T = T_k \propto (\theta_1 - \theta_2)$$

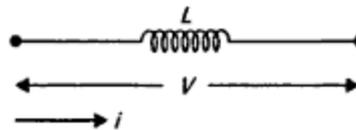
## Modeling of electrical system

- Electrical circuits involving resistors, capacitors and inductors are considered. The behavior of such systems is governed by Ohm's law and Kirchhoff's laws.
- Resistor: Consider a resistance of 'R' carrying current 'I' Amps as shown in Fig (a), then the voltage drop across it is  $v = R I$



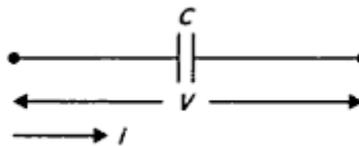
- Inductor: Consider an inductor "L' H carrying current 'I' Amps as shown in Fig (a), then the voltage drop across it can be written as

$$V = L \frac{di}{dt}$$



- Capacitor: Consider a capacitor "C' F carrying current "I' Amps as shown in Fig (a), then the voltage drop across it can be written as

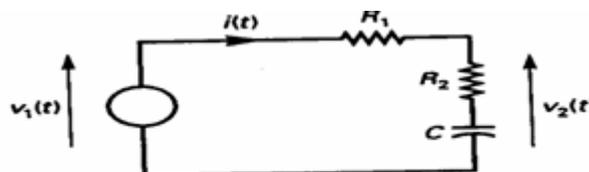
$$V = \frac{1}{C} \int i dt$$



## Steps for modeling of electrical system

- Apply Kirchhoff's voltage law or Kirchhoff's current law to form the differential equations describing electrical circuits comprising of resistors, capacitors, and inductors.
- Form Transfer Functions from the describing differential equations.
- Then simulate the model.

### Example



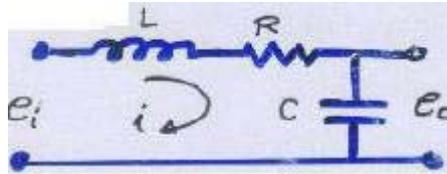
$$R_1 i(t) + R_2 i(t) + \frac{1}{C} \int_0^t i dt = V_1(t)$$

$$R_2 i(t) + \frac{1}{C} \int_0^t i dt = V_2(t)$$

## Electrical systems

LRC circuit. Applying Kirchoff's voltage law to the system shown. We obtain the following equation

### Resistance circuit



$$L \frac{di}{dt} + Ri + \frac{1}{C} \int i dt = e_i \quad (1)$$

$$\frac{1}{C} \int i dt = e_o \quad (2)$$

Equation (1) & (2) give a mathematical model of the circuit. Taking the L.T. of equations (1) & (2), assuming zero initial conditions, we obtain

$$LsI(s) + RI(s) + \frac{1}{Cs}I(s) = E_i(s)$$

$$\frac{1}{Cs}I(s) = E_o(s)$$

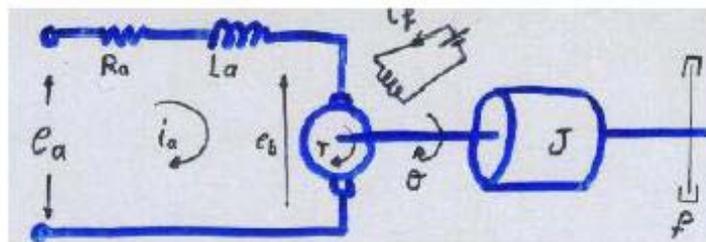
$$\frac{E_o(s)}{E_i(s)} = \frac{1}{LCs^2 + RCs + 1}$$

The transfer function

### Armature-Controlled dc motors

The dc motors have separately excited fields. They are either armature-controlled with fixed field or field-controlled with fixed armature current. For example, dc motors used in instruments employ a fixed permanent-magnet field, and the controlled signal is applied to the armature terminals.

Consider the armature-controlled dc motor shown in the following figure



$R_a$  = armature-winding resistance, ohms

$L_a$  = armature-winding inductance, henrys

$I_a$  = armature-winding current, amperes

$I_f$  = field current, amperes

$E_a$  = applied armature voltage, volt

$E_b$  = back emf, volts

$\theta$  = angular displacement of the motor shaft, radians

$T$  = torque delivered by the motor, Newton\*meter

$J$  = equivalent moment of inertia of the motor and load referred to the motor shaft  $kg \cdot m^2$

$f$  = equivalent viscous-friction coefficient of the motor and load referred to the motor shaft. Newton\*m/rad/s

$T = k_1 I_a \psi$  where  $\psi$  is the air gap flux,  $\psi = k_f I_f$ ,  $k_1$  is constant

For the constant flux

$$e_b = k_b \frac{d\theta}{dt} \quad (1)$$

Where  $K_b$  is a back emf constant

The differential equation for the armature circuit

$$L_a \frac{di_a}{dt} + R_a i_a + e_b = e_a \quad (2)$$

The armature current produces the torque which is applied to the inertia and friction; hence

$$J \frac{d^2\theta}{dt^2} + f \frac{d\theta}{dt} = T = K i_a \quad (3)$$

Assuming that all initial conditions are condition are zero/and taking the L.T. of equations (1), (2) & (3), we obtain

$$K_b s \theta(s) = E_b(s)$$

$$(L_a s + R_a) i_a(s) + E_b(s) = E_a(s) (J s^2 + f s)$$

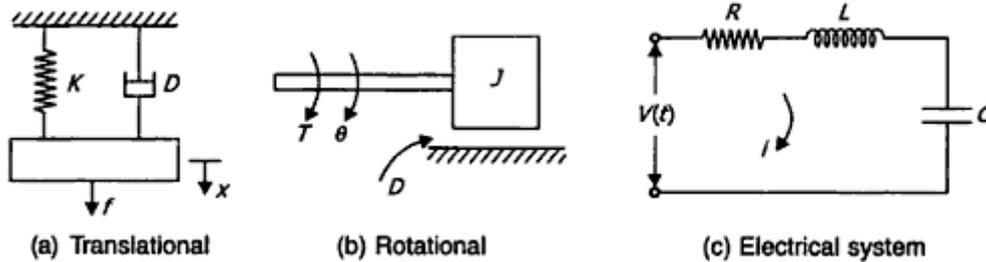
$$\theta(s) = T(s) = K i_a(s)$$

The T.F can be obtained is

$$\frac{\theta(s)}{E_a(s)} = \frac{K}{s(L_a J s^2 + (L_a f + R_a J) s + R_a f + K K_b)}$$

### Analogous Systems

Let us consider a mechanical (both translational and rotational) and electrical system as shown in the fig.



From the fig (a)

We get  $M \frac{d^2x}{dt^2} + D \frac{dx}{dt} + Kx = F$  (1)

From the fig (b)

We get  $J \frac{d^2\theta}{dt^2} + D \frac{d\theta}{dt} + K\theta = T$  (2)

From the fig (c)

We get  $L \frac{d^2q}{dt^2} + R \frac{dq}{dt} + q \frac{1}{C} = v(t)$  (3)

Where  $q = \int i dt$

They are two methods to get analogous system. These are (i) force- voltage (f-v) analogy and (ii) force-current (f-c) analogy

#### (i) Force – Voltage (f-v) Analogy

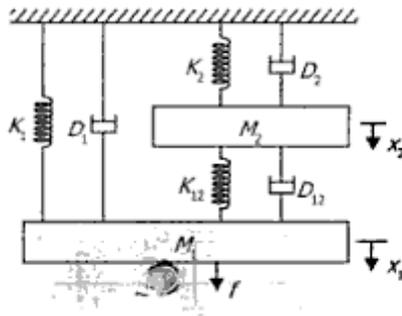
Translational	Electrical	Rotational
Force (F)	Voltage (V)	Torque (T)
Mass (M)	Inductance (L)	Inertia (J)
Damper (D)	Resistance (R)	Damper (D)
Spring (K)	Elastance (1/C)	Spring (K)
Displacement (x)	Charge (q)	Displacement ( $\theta$ )
Velocity (u)	Current (I)	Velocity ( $\omega$ )

#### (ii) Force – Current (f-c) Analogy

Translational	Electrical	Rotational
Force (F)	Current (I)	Torque (T)
Mass (M)	Capacitance (C)	Inertia (J)
Damper (D)	Reciprocal of Inductance (1/L)	Damper (D)
Spring (K)	Conductance (1/K)	Spring (K)
Displacement (x)	Flux Linkage ( $\psi$ )	Displacement ( $\theta$ )
Velocity ( $u - \frac{dx}{dt}$ )	Voltage ( $v - \frac{d\psi}{dt}$ )	Velocity ( $\omega - \frac{d\theta}{dt}$ )

**Problem**

Find the system equation for system shown in the fig. And also determine f-v and f-i analogies



For free body diagram M1

$$f = M_1 \frac{d^2 x_1}{dt^2} + D_1 \frac{dx_1}{dt} + K_1 x_1 + D_{12} \frac{d}{dt}(x_1 - x_2) + K_{12} (x_1 - x_2) \quad (1)$$

For free body diagram M2

$$K_{12}(x_1 - x_2) + D_{12} \frac{d}{dt}(x_1 - x_2) = M_2 \frac{d^2 x_2}{dt^2} + D_2 \frac{dx_2}{dt} + K_2 x_2 \quad (2)$$

Force –voltage analogy

$$f \rightarrow v, M \rightarrow L, D \rightarrow R, K \rightarrow \frac{1}{C}, x \rightarrow q$$

From Eq. (1) we get

$$v = L_1 \frac{d^2 q_1}{dt^2} + R_1 \frac{dq_1}{dt} + \frac{1}{C_1} q_1 + R_{12} \frac{d}{dt}(q_1 - q_2) + \frac{1}{C_{12}(q_1 - q_2)} \quad (3)$$

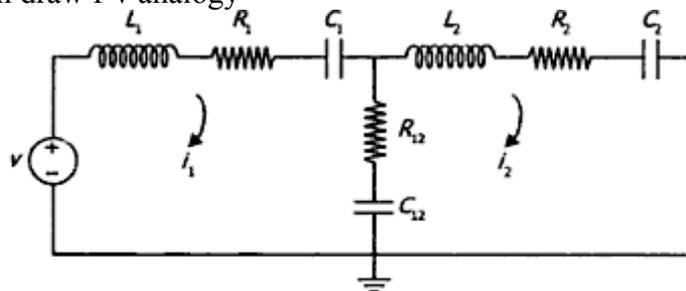
$$v = L_1 \frac{dt_1}{dt} + R_1 t_1 + \frac{1}{C_1 \int t_1 dt} + R_{12}(t_1 - t_2) + \frac{1}{C_{12} \int (t_1 - t_2) dt}$$

From Eq. (2) we get

$$\frac{1}{C_{12}}(q_1 - q_2) + R_{12} \frac{d}{dt}(q_1 - q_2) = L_2 \frac{d^2 q_2}{dt^2} + R_2 \frac{dq_2}{dt} + \frac{1}{C_2} q_2 \quad (4)$$

$$\frac{1}{C_{12} \int (t_1 - t_2) dt} + R_{12}(t_1 - t_2) = L_2 \frac{dt_2}{dt} + R_2 t_2 + \frac{1}{C_2 \int t_2 dt}$$

From Eq. (3) and (4) we can draw f-v analogy



Force–current analogy

$$f \rightarrow i, M \rightarrow C, D \rightarrow \frac{1}{R}, K \rightarrow \frac{1}{L}, x \rightarrow \psi$$

From Eq. (1) we get

$$i = C_1 \frac{d^2 \psi_1}{dt^2} + \frac{1}{R_1} \frac{d\psi_1}{dt} + \frac{1}{L_1} \psi_1 + \frac{1}{R_{12}} \frac{d}{dt} (\psi_1 - \psi_2) + \frac{1}{L_{12}} (\psi_1 - \psi_2)$$

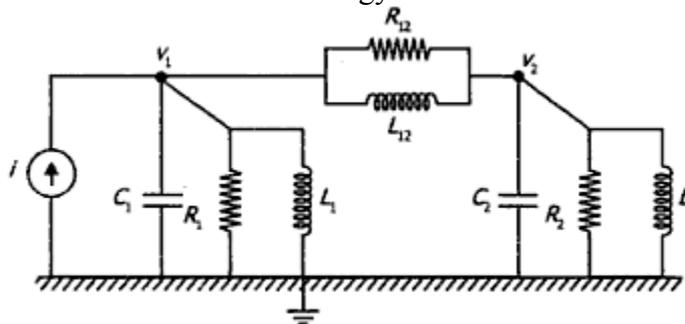
$$i = C_1 \frac{dv_1}{dt} + \frac{1}{R_1} v_1 + \frac{1}{L_1} \int i_1 dt + \frac{(v_1 - v_2)}{R_{12}} + \frac{1}{L_{12}} \int (v_1 - v_2) dt \quad (5)$$

From eq (2) we get

$$\frac{1}{L_{12}} (\psi_1 - \psi_2) + \frac{1}{R_{12}} \frac{d}{dt} (\psi_1 - \psi_2) = C_2 \frac{d^2 \psi_2}{dt^2} + \frac{1}{R_2} \frac{d\psi_2}{dt} + \frac{1}{L_2} \psi_2$$

$$\frac{1}{L_{12}} \int (v_1 - v_2) dt + \frac{1}{R_{12}} (v_1 - v_2) = C_2 \frac{dv_2}{dt} + \frac{v_2}{R_2} + \frac{1}{L_2} \int v_2 dt \quad (6)$$

From eq (5) and (6) we can draw force-current analogy



The system can be represented in two forms:

- Block diagram representation
- Signal flow graph

### Block diagram

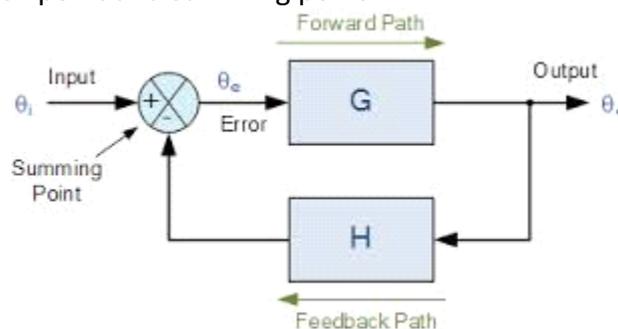
A pictorial representation of the functions performed by each component and of the flow of signals

#### Basic elements of a block diagram

- Blocks
- Transfer functions of elements inside the blocks
- Summing points
- Take off points
- Arrow

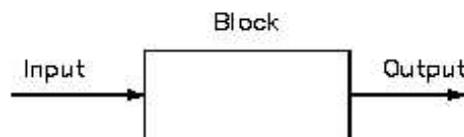
### Block diagram

A control system may consist of a number of components. A block diagram of a system is a pictorial representation of the functions performed by each component and of the flow of signals. The elements of a block diagram are block, branch point and summing point.



### Block

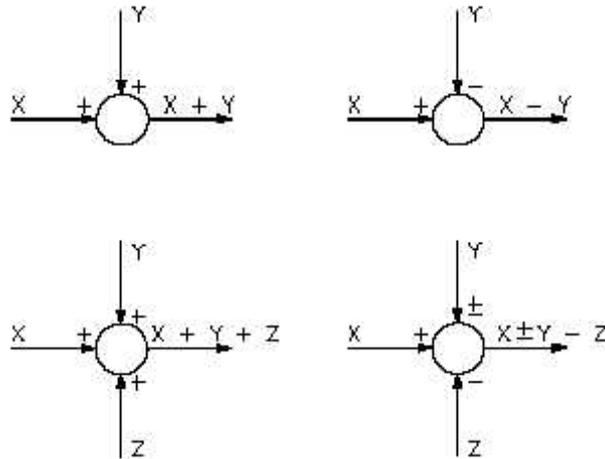
In a block diagram all system variables are linked to each other through functional blocks. The functional block or simply block is a symbol for the mathematical operation on the input signal to the block that produces the output.



### Summing point

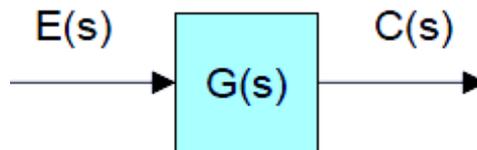
Although blocks are used to identify many types of mathematical operations, operations of addition and subtraction are represented by a circle, called a summing point. As shown in Figure a summing point may have one or several inputs. Each input has its own appropriate plus or minus sign.

A summing point has only one output and is equal to the algebraic sum of the inputs.



A takeoff point is used to allow a signal to be used by more than one block or summing point. The transfer function is given inside the block

- The input in this case is  $E(s)$
- The output in this case is  $C(s)$
- $C(s) = G(s) E(s)$



- **Functional block** – each element of the practical system represented by block with its
- **Branches** – lines showing the connection between the blocks
- **Arrow** – associated with each branch to indicate the direction of flow of signal
- **Closed loop system**
- **Summing point** – comparing the different signals
- **Take off point** – point from which signal is taken for feed back

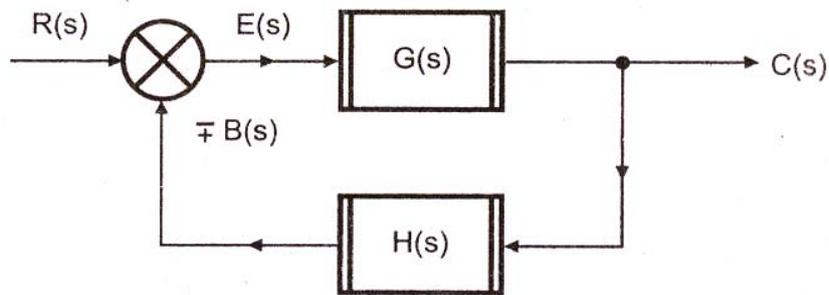
### Advantages of Block Diagram Representation

- Very simple to construct block diagram for a complicated system
- Function of individual element can be visualized
- Individual & Overall performance can be studied
- Over all transfer function can be calculated easily

### Disadvantages of Block Diagram Representation

- No information about the physical construction
- Source of energy is not shown

## Simple or Canonical form of closed loop system

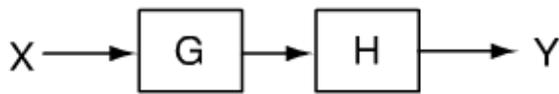


- $R(s)$  – Laplace of reference input  $r(t)$
- $C(s)$  – Laplace of controlled output  $c(t)$
- $E(s)$  – Laplace of error signal  $e(t)$
- $B(s)$  – Laplace of feedback signal  $b(t)$
- $G(s)$  – Forward path transfer function
- $H(s)$  – Feedback path transfer function

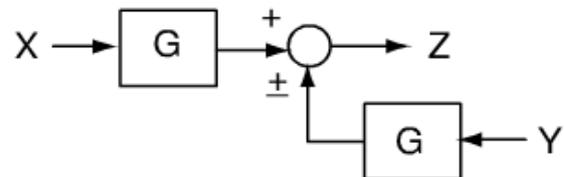
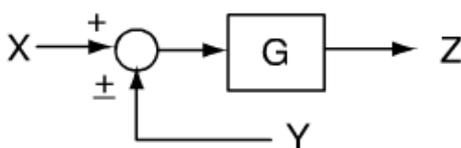
### Block diagram reduction technique

Because of their simplicity and versatility, block diagrams are often used by control engineers to describe all types of systems. A block diagram can be used simply to represent the composition and interconnection of a system. Also, it can be used, together with transfer functions, to represent the cause-and-effect relationships throughout the system. Transfer Function is defined as the relationship between an input signal and an output signal to a device

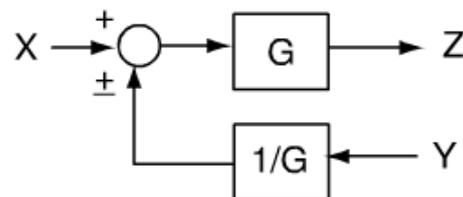
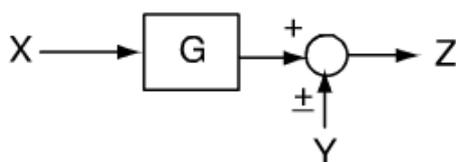
### Block diagram rules



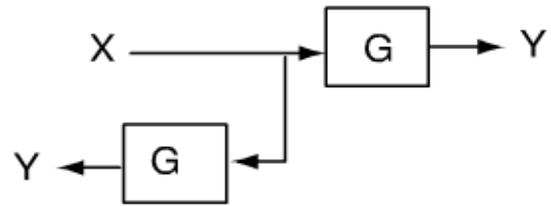
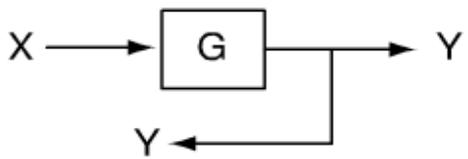
Cascaded blocks



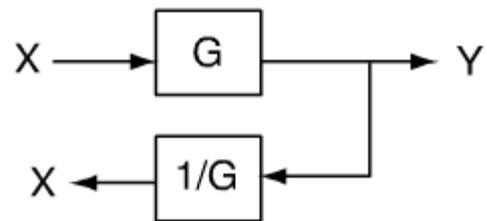
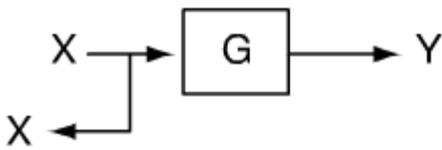
Moving a summer beyond the block moving



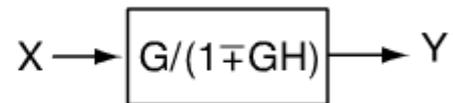
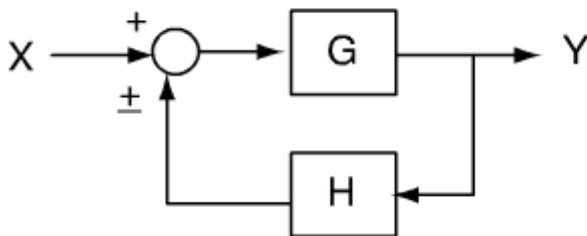
Moving a summer ahead of block



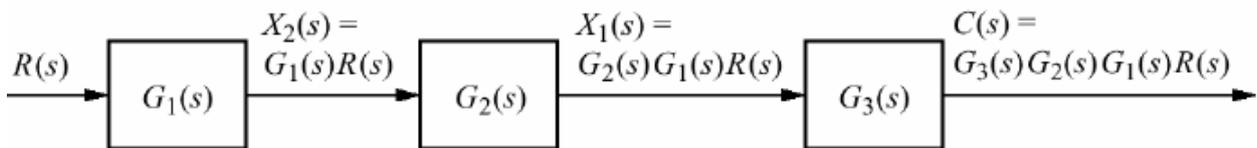
Moving a pick-off ahead of block



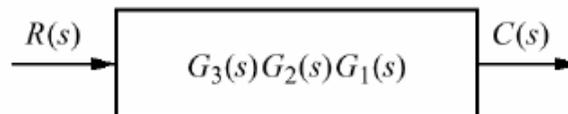
Moving a pick-off beyond a block



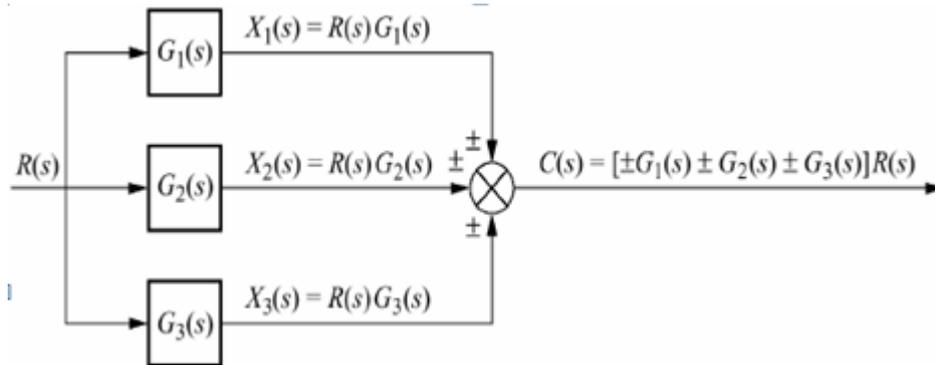
Eliminating a feedback loop



(a)

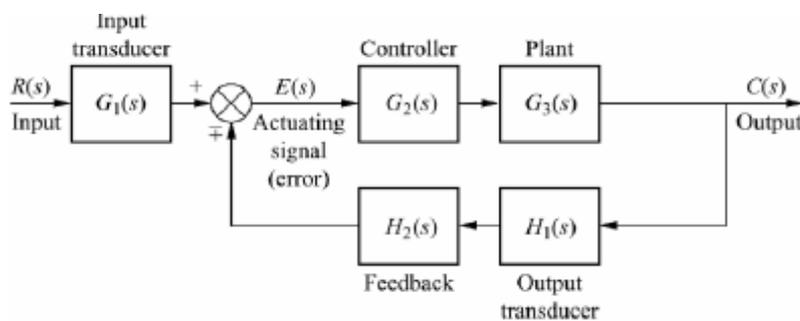


Cascaded Subsystems



Parallel Subsystems

## Feedback Control System

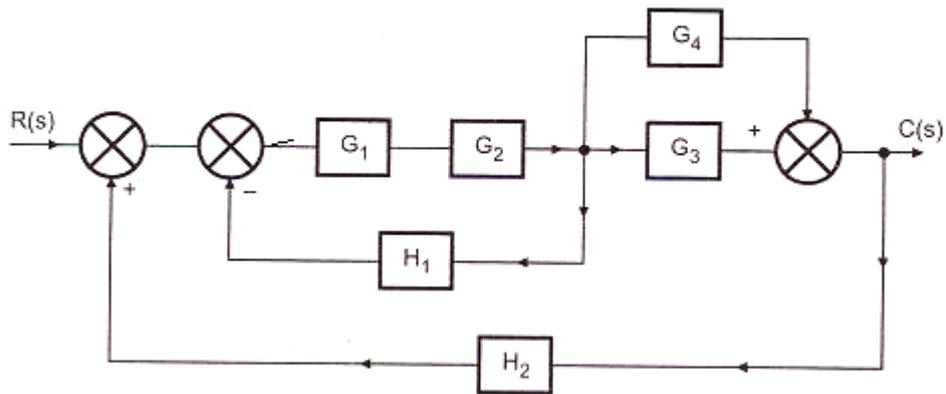


### Procedure to solve Block Diagram Reduction Problems

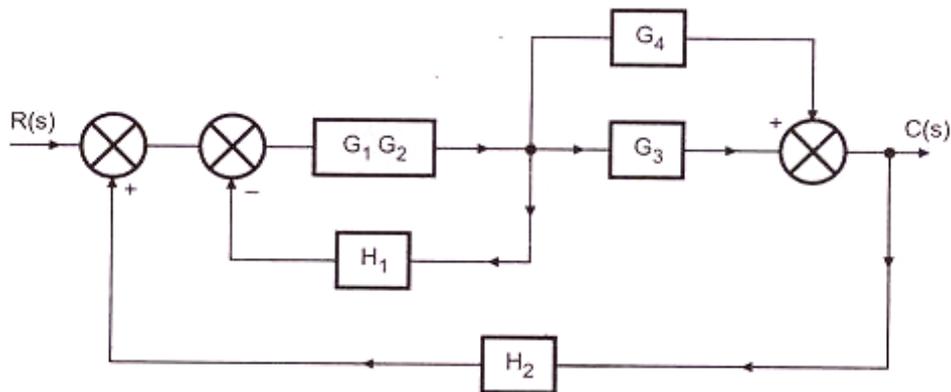
- Step 1: Reduce the blocks connected in series
- Step 2: Reduce the blocks connected in parallel
- Step 3: Reduce the minor feedback loops
- Step 4: Try to shift take off points towards right and Summing point towards left
- Step 5: Repeat steps 1 to 4 till simple form is obtained
- Step 6: Obtain the Transfer Function of Overall System

## Problem 1

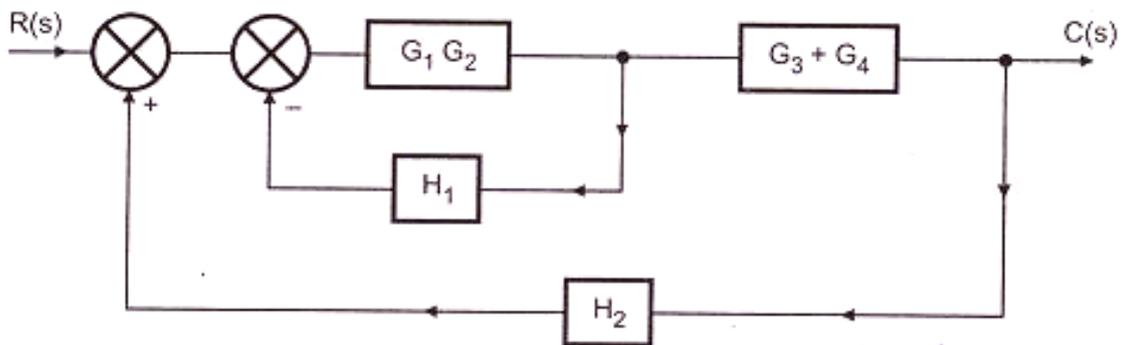
Obtain the Transfer function of the given block diagram



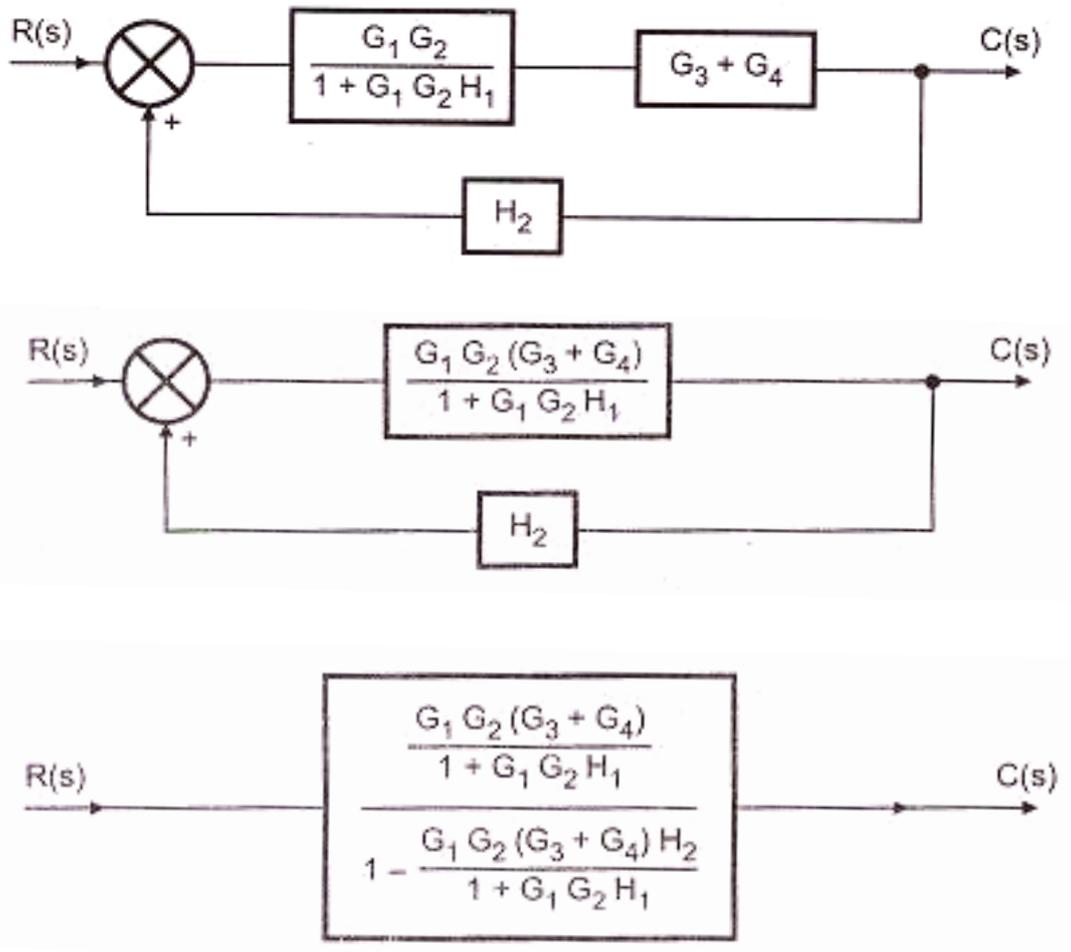
Combine  $G_1, G_2$  which are in series



Combine  $G_3, G_4$  which are in Parallel



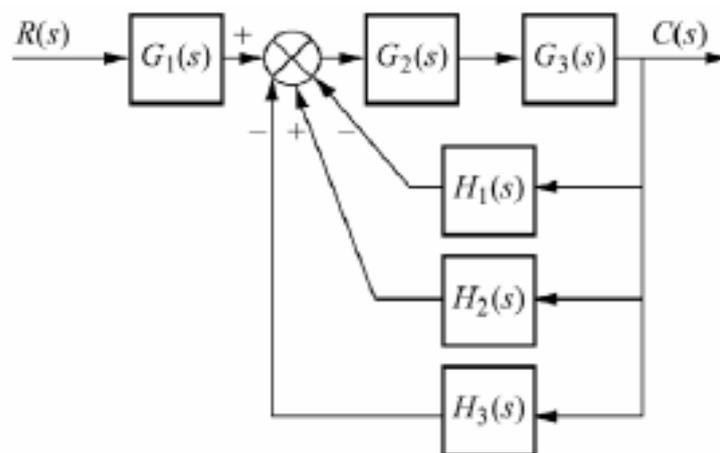
Reduce minor feedback loop of  $G_1$ ,  $G_2$  and  $H_1$

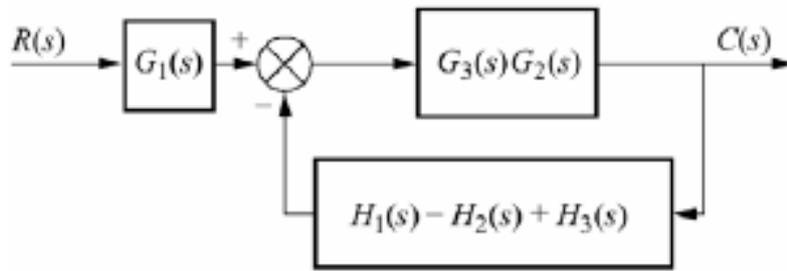


Transfer function

$$\frac{C(s)}{R(s)} = \frac{G_1 G_2 (G_3 + G_4)}{1 + G_1 G_2 H_1 - G_1 G_2 (G_3 + G_4) H_2}$$

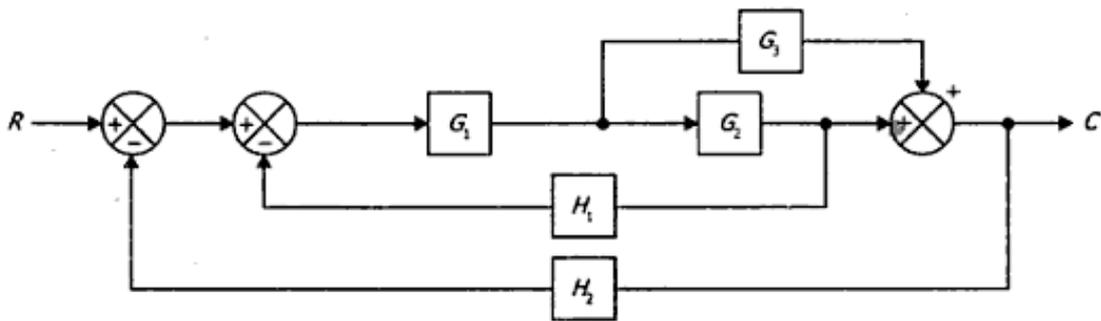
2. Obtain the transfer function for the system shown in the fig



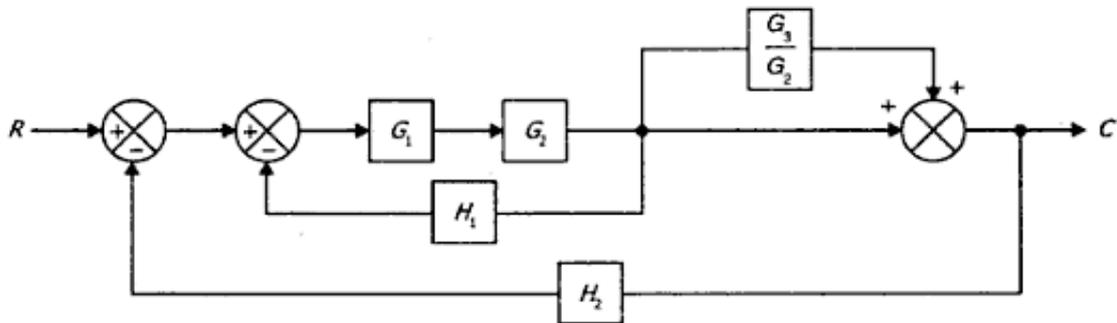


$$R(s) \rightarrow \frac{G_3(s)G_2(s)G_1(s)}{1 + G_3(s)G_2(s)[H_1(s) - H_2(s) + H_3(s)]} C(s)$$

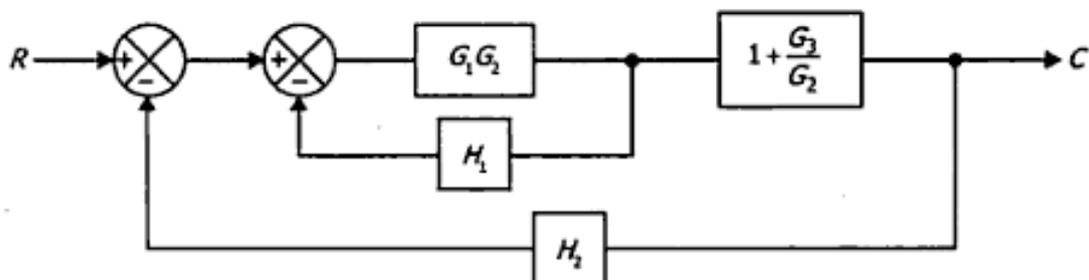
3. Obtain the transfer function  $C/R$  for the block diagram shown in the fig



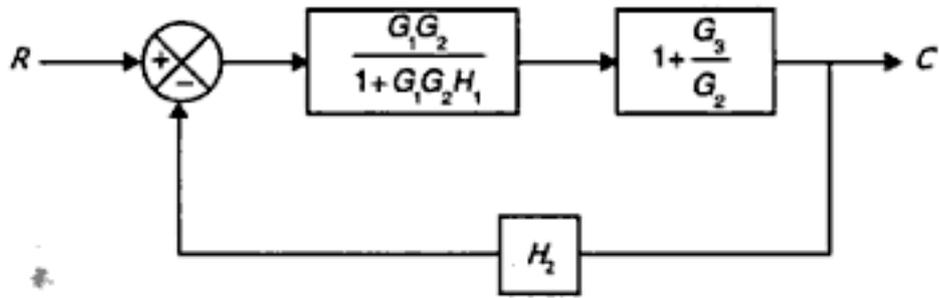
The take-off point is shifted after the block  $G_2$



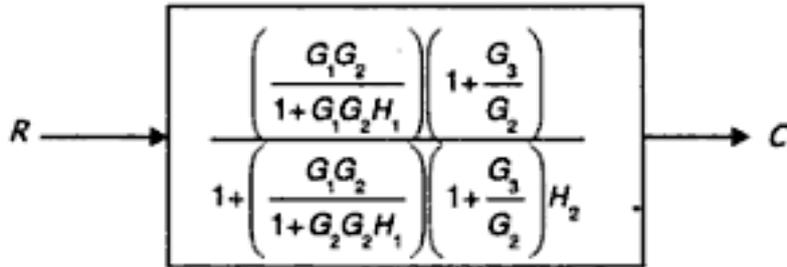
Reducing the cascade block and parallel block



Replacing the internal feedback loop



Equivalent block diagram



Transfer function

$$\frac{C}{R} = \frac{\frac{G_1 G_2 (G_2 + G_3)}{1 + G_1 G_2 H_1}}{1 + \frac{G_1 (G_2 + G_3) H_2}{1 + G_1 G_2 H_1}}$$

$$\frac{G_1 (G_2 + G_3)}{1 + G_1 G_2 (H_1 + H_2) + G_1 G_3 H_2}$$

### Signal Flow Graph Representation

Signal Flow Graph Representation of a system obtained from the equations, which shows the flow of the signal

### Signal flow graph

A signal flow graph is a diagram that represents a set of simultaneous linear algebraic equations. By taking Laplace transfer, the time domain differential equations governing a control system can be transferred to a set of algebraic equation in s-domain. A signal-flow graph consists of a network in which nodes are connected by directed branches. It depicts the flow of signals from one point of a system to another and gives the relationships among the signals.

### Basic Elements of a Signal flow graph

- **Node** - a point representing a signal or variable.
- **Branch** – unidirectional line segment joining two nodes.
- **Path** – a branch or a continuous sequence of branches that can be traversed from one node to another node.
- **Loop** – a closed path that originates and terminates on the same node and along the path no node is met twice.
- **Nonteaching loops** – two loops are said to be non touching if they do not have a common node.

### Mason's gain formula

The relationship between an input variable and an output variable of signal flow graph is given by the net gain between the input and the output nodes is known as overall gain of the system. Mason's gain rule for the determination of the overall system gain is given below.

$$M = \frac{1}{\Delta} \sum_{k=1}^N P_k \Delta_k = \frac{X_{out}}{X_{in}}$$

Where  $M$  = gain between  $X_{in}$  and  $X_{out}$

$X_{out}$  = output node variable

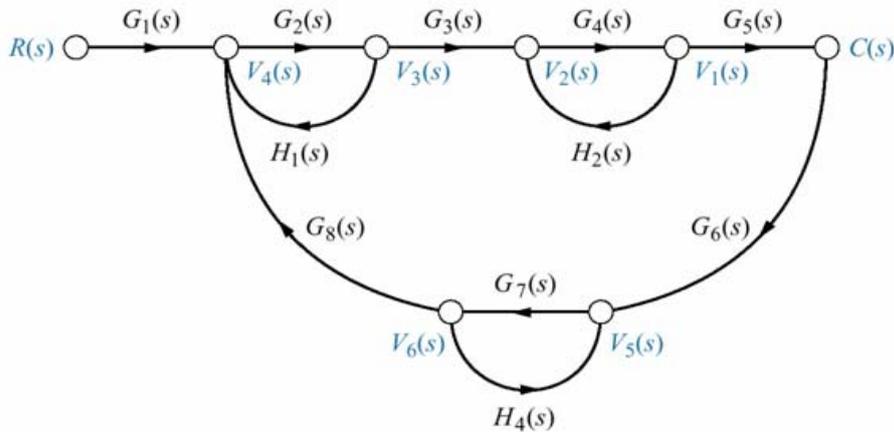
$X_{in}$  = input node variable

$N$  = total number of forward paths

$P_k$  = path gain of the  $k$  th forward path

$\Delta = 1 - (\text{sum of loop gains of all individual loop}) + (\text{sum of gain product of all possible combinations of two non touching loops}) - (\text{sum of gain products of all possible combination of three non touching loops})$

## Problem



Forward path gain:  $T_1 = G_1(s)G_2(s)G_3(s)G_4(s)G_5(s)$

Closed loop gain

1.  $G_2(s)H_1(s)$
2.  $G_4(s)H_2(s)$
3.  $G_7(s)H_4(s)$
4.  $G_2(s)G_3(s)G_4(s)G_5(s)G_6(s)G_7(s)G_8(s)$

Non touching loops taken two at a time

5. Loop 1 and loop 2:  $G_2(s)H_1(s)G_4(s)H_2(s)$
6. Loop 1 and loop 3:  $G_2(s)H_1(s)G_7(s)H_4(s)$
7. Loop 2 and loop 3:  $G_4(s)H_2(s)G_7(s)H_4(s)$

Non touching loops taken three at a time

8. Loop 1, 2, 3:  $G_2(s)H_1(s)G_4(s)H_2(s)G_7(s)H_4(s)$

Now,  $\Delta = 1 - \{ (1) + (2) + (3) + (4) \} + \{ (5) + (6) + (7) \} - (8)$

Portion of  $\Delta$  not touching the forward path

$$\Delta_1 = 1 - G_7(s)H_4(s)$$

Hence,

$$C(s) = \frac{C(s)}{R(s)} = \frac{T_1 \Delta_1}{\Delta}$$

$$= \frac{G_1(s)G_2(s)G_3(s)G_4(s)G_5(s)[1 - G_7(s)H_4(s)]}{\Delta}$$

### 3. Feedback characteristics of Control Systems:

In control system, the feedback reduces the error, also reduces the sensitivity of the system to parameter variations. The parameter may vary due to some change in conditions. The variation in parameter affects the performance of the system. So it is necessary to make the system insensitive to such parameter variations.

#### Effect of feedback on sensitivity

The parameters of any control system changes with the change environment conditions. Also these parameters cannot be constant throughout the life. These parameter variations affect the performance of the system. For example, the resistance of winding of a motor changes due to change in temperature during its operation.

- Sensitivity to model uncertainties

$$S_G^T = \frac{\text{Ratio of \% change in sys T.F.}}{\text{Ratio of \% change in process T.F.}}$$

$$\frac{\frac{\Delta T}{T}}{\frac{\Delta G}{G}} = \frac{\partial \ln T}{\partial \ln G} = \frac{\partial T}{\partial G} \frac{G}{T}$$

#### Open loop:

$$\Delta Y(s) = \Delta G(s)R(s), \quad \Delta T(s) = \frac{\Delta Y(s)}{R(s)} = \Delta G(s)$$

#### Closed-loop:

$$T(s) = \frac{G}{1 + GH}, \quad \frac{\partial T}{\partial G} = \frac{(1 + GH) - GH}{(1 + GH)^2} = \frac{1}{(1 + GH)^2}$$

$$S_G^T = \frac{\partial T}{\partial G} \frac{G}{T} = \frac{1}{(1 + GH)^2} \frac{G}{\frac{G}{1 + GH}} = \frac{1}{(1 + GH)}$$

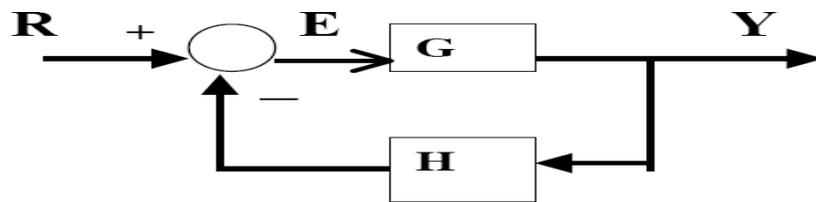
→ Reduced  $S_G^T$  below that of the open-loop sys by increasing  $G*H$  ( $\gg 1.0$ ).

$$S_H^T = \frac{\partial T}{\partial H} \frac{H}{T} = \frac{-GH}{1 + GH}$$

\* if  $GH \gg 1.0 \rightarrow S_H^T = -1$

Feedback components should not be varied with environmental changes → change in  $H(s)$  directly affects output response

Find transfer function of a feedback control system -



### Closed- loop control

- Has ability to reduce system sensitivity
- If  $G(s)H(s) \gg 1$  for all complex frequency of interest, then:

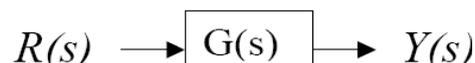
$$Y(s) = \frac{G}{1 + GH} R \Rightarrow \frac{G}{GH} R = \frac{1}{H} R$$

- By increasing the gain of  $G(s)H(s)$  it reduces the effect of  $G(s)$  on the input  $\rightarrow$  variation of the parameters of the process,  $G(s)$ , is reduced
- But, making  $G(s)H(s) \gg 1$  can lead to highly oscillatory and even unstable response

### When – Process, $G(s)$ , is changed

Open loop –

$$Y(s) = \Delta G(s)R(s)$$



Closed loop –

$$Y(s) + \Delta Y(s) = \frac{G(s) + \Delta G(s)}{1 + (G(s) + \Delta G(s))H(s)} R(s)$$

$$Y = \frac{G}{1 + GH} R$$

Then the change in the output is

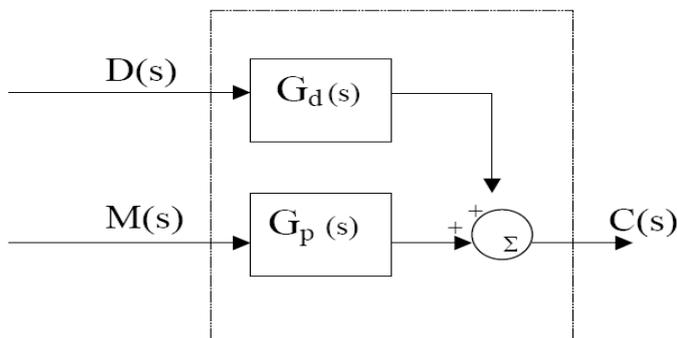
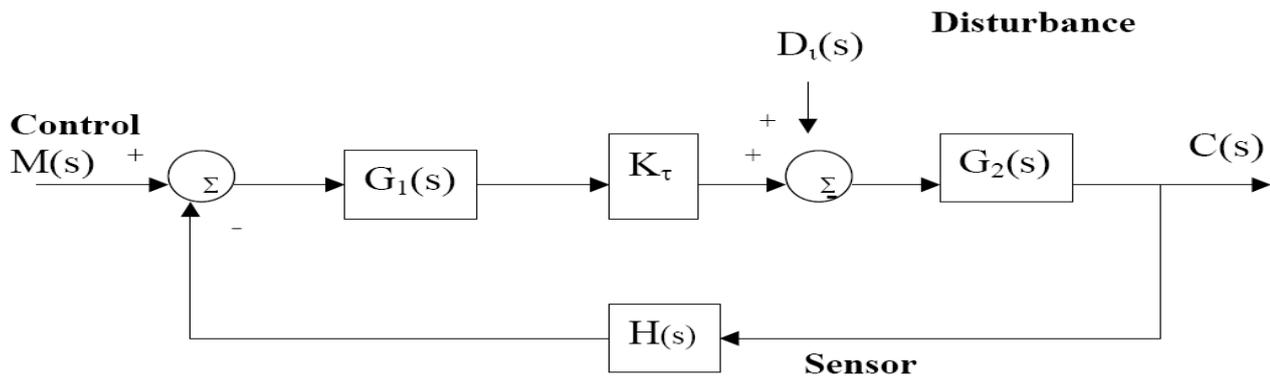
$$\Delta Y(s) = \frac{\Delta G(s)}{(1 + GH(s) + \Delta GH(s))(1 + GH(s))} R(s)$$

when  $GH \gg \Delta GH(s)$ , as is often the case, we have

$$\Delta Y(s) = \frac{\Delta G(s)}{(1 + GH(s))^2} R(s)$$

Change of the output is reduced by  $[1 + GH]$

## Disturbance in a system



**By superposition**

$$\begin{aligned} C(s) &= \frac{K_\tau G_1 G_2}{1 + K_\tau G_1 G_2 H} M(s) \\ &+ \frac{G_2}{1 + K_\tau G_1 G_2 H} D\tau(s) \\ &= G_p(s)M(s) + G_d(s)D\tau(s) \end{aligned}$$

### State Variable Model

$$\dot{X} = Ax + Bu$$

$$y = Cx$$

$$u = \begin{bmatrix} m(t) \\ d(t) \end{bmatrix}$$

The transfer function  $G(s) = C[sI - A]^{-1}B$

$$= [G_p(s)]$$

T.F from the control Input

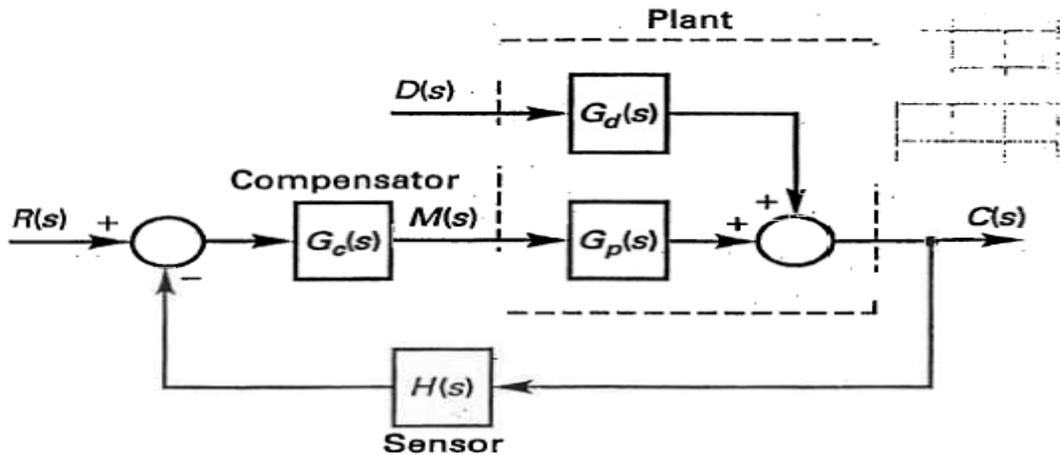
$$[ \quad G_d(s) ]$$

T.F from D(s) to the

M(s) to the output

output

## Disturbance in a closed-loop system:



Where

$$C(s) = \left[ \frac{G_c G_p}{1 + G_c G_p H} \right] R(s) + \left[ \frac{G_d}{1 + G_c G_p H} \right] D(s)$$

$$= T(s)R(s) + T_d(s)D(s)$$

The loop gain  $G_c G_p H$  must be made large to reduce the system sensitivity

$$T_d = \frac{G_d}{1 + G_c G_p H} = \frac{G_d}{G_c G_p H}$$

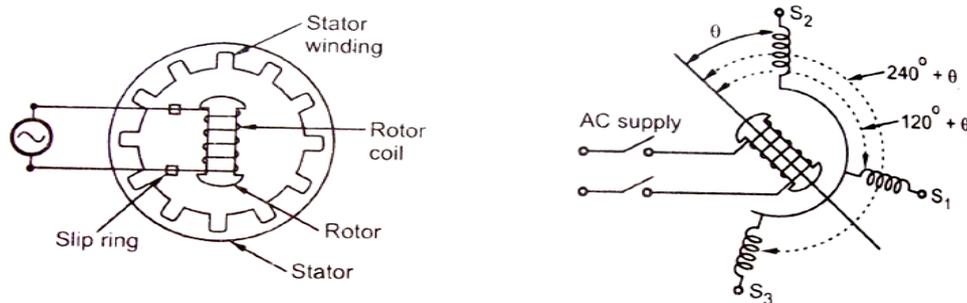
### Reducing Disturbance

- Reduce the gain  $G_d(s)$
- Increase the loop gain  $G_c G_p H$  (Choice of  $G_c$ )
- Reduce the disturbance  $d(t)$
- Feed forward method if the disturbance can be measured

#### 4.Control Components:

##### SYNCHROS:

A synchro is a type of rotary electrical transformer that is used for measuring the angle of a rotating machine such as an antenna platform. In its general physical construction, it is much like an electric motor. The primary winding of the transformer, fixed to the rotor, is excited by an alternating current, which by electromagnetic induction, causes currents to flow in three star-connected secondary windings fixed at 120 degrees to each other on the stator. The relative magnitudes of secondary currents are measured and used to determine the angle of the rotor relative to the stator, or the currents can be used to directly drive a receiver synchro that will rotate in unison with the synchro transmitter



##### Synchro Operation:

On a practical level, Synchro resembles motors, in that there is a rotor, stator, and a shaft. Ordinarily, slip rings and brushes connect the rotor to external power. A synchro transmitter's shaft is rotated by the mechanism that sends information, while the synchro receiver's shaft rotates a dial, or operates a light mechanical load. Single and three-phase units are common in use, and will follow the other's rotation when connected properly. One transmitter can turn several receivers; if torque is a factor, the transmitter must be physically larger to source the additional current.

##### Uses of Synchro:

- Synchro systems were first used in the control system of the Panama Canal in the early 1900s to transmit lock gate and valve stem positions and water levels to the control desks
- Fire-control system designs developed during World War II used synchros extensively, to transmit angular information from guns and sights to an analog fire control computer, and to transmit the desired gun position back to the gun location.

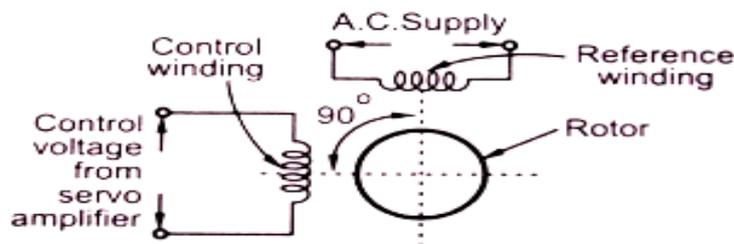
## Tachometers

Tachometer is an electromechanical unit which generates an electrical output proportional to the speed of the shaft. In automatic control system tachometer performs two main functions:

- Stabilization of system
- Computation of closed loops in a control system

ACTachometer:

The AC tachometer is a device, which is similar to a two phase induction motor, in which two stator windings are placed in quadrature with each other and rotor is short circuited. In AC Tachometer, a sinusoidal voltage of rated value is applied to the primary winding, which is known as reference winding, the secondary winding is placed 90 degrees apart from primary winding. The magnitude of sinusoidal output voltage is directly proportional to the speed of rotor.



## D.C. Tachometer

In control systems most common type of tachometers are d.c. tachometers .D.C. Tachometer contains an iron core rotor and permanent magnet. The magnetic field is provided by the permanent magnet and no external supply voltage is necessary. The input to the tachometer is the speed of the shaft and the output is voltage which is proportional to the angular speed of the shaft.

$$e = K \omega(t)$$

Where  $e$  = tachometer generator voltage  
 $K$  =tachometer sensitivity  
 $\omega$  = angular speed of shaft

Laplace transform of equation,

$$E(s) = K\omega(s)$$

Hence transfer function of tachometer is

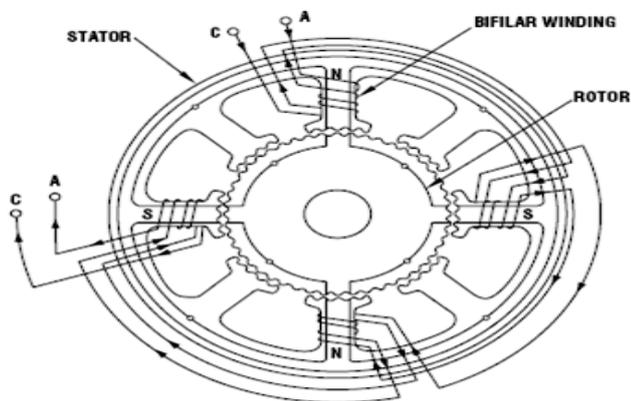
$$K = \frac{E(S)}{\omega(S)}$$

In d.c tachometer the winding on rotor are connected to the commutator and the output voltage is taken across the brushes. The permanent magnet tachometers are compact and reliable but having high inertia. For reducing the voltage drop across the brushes, metal brushes with silver tips are used.

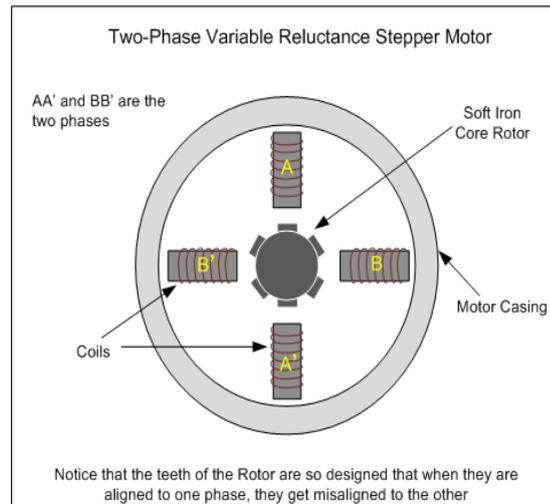
#### Stepper Motors:

In stepper motors, the movement of rotor is in discrete steps. A stepper motor is electromechanical device. There are three types of stepper motors.

1. Variable reluctance motors
2. Permanent magnet motors
3. Hybrid type



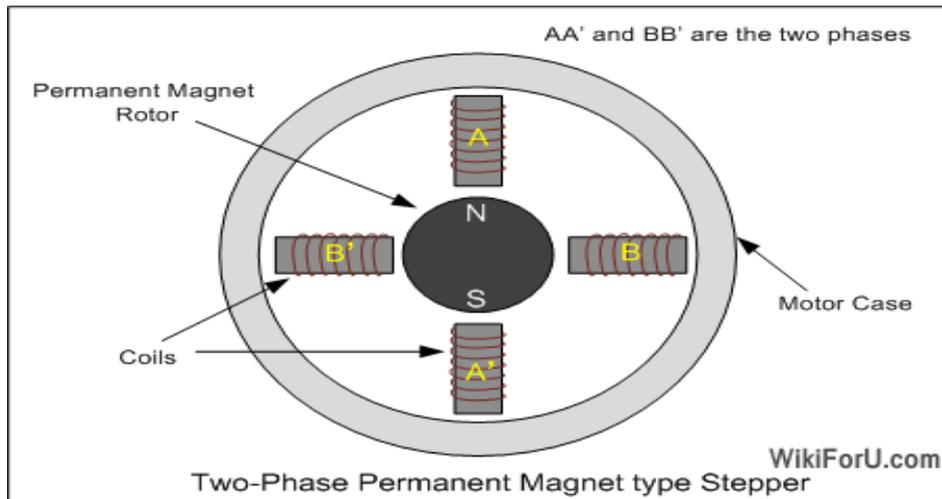
1.8° Hybrid motor.



- Conventional servo motors are classified as continuous rotation motors
- Stepper motors rotate through a specific number of degrees, or steps, then stop
- Each incoming pulse results in the shaft turning a specific angular distance
- Stepper motors can control velocity, distance, and direction of mechanical load

#### Permanent Magnet Stepper Motor:

- PM stepper motors have rotor teeth made of permanent magnets
- Reaction of the rotor teeth to stator fields provides torque for the motor.
- Signals are applied to the stator to determine direction and step rate of the rotor.



## Stepper Motor Speed

Stepper motor speed depends upon the step angle and stepping rate

$$N = (Y \times S) / 6$$

Where  $n$  = speed in RPM

$Y$  = step angle in degrees

$S$  = Steps per second

$6$  = Formula constant

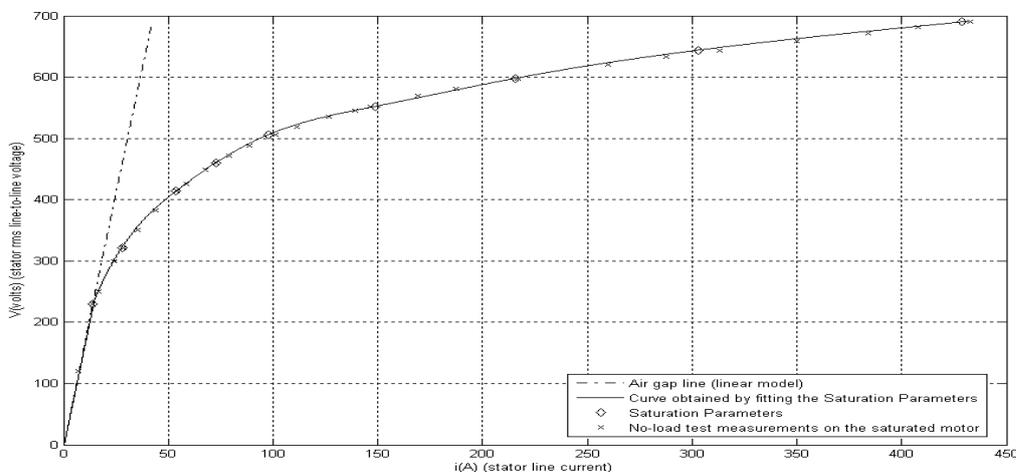
## Amplidyne:-

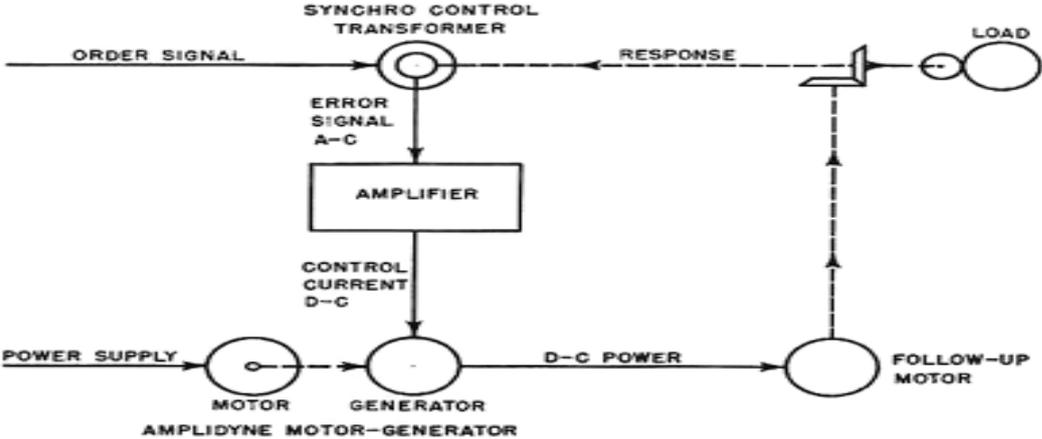
An Amplidyne is a rotating amplifier. It is a prime-mover-driven d.c. generator whose output power can be controlled by a small field power input. An amplidyne is capable of giving a controlled power output in the range of a few hundred to few thousand watts with a power amplification of the order of 10,000 or more and hence finds wide application in feedback control system.

Study of amplidyne whose output power can be controlled by a small field power input. For studying the characteristics.

(i) We plot a graph of Output voltage against effective field current with no load, full load (500W), and without compensation winding effect

(ii) We draw the schematic diagram of an amplidyne system





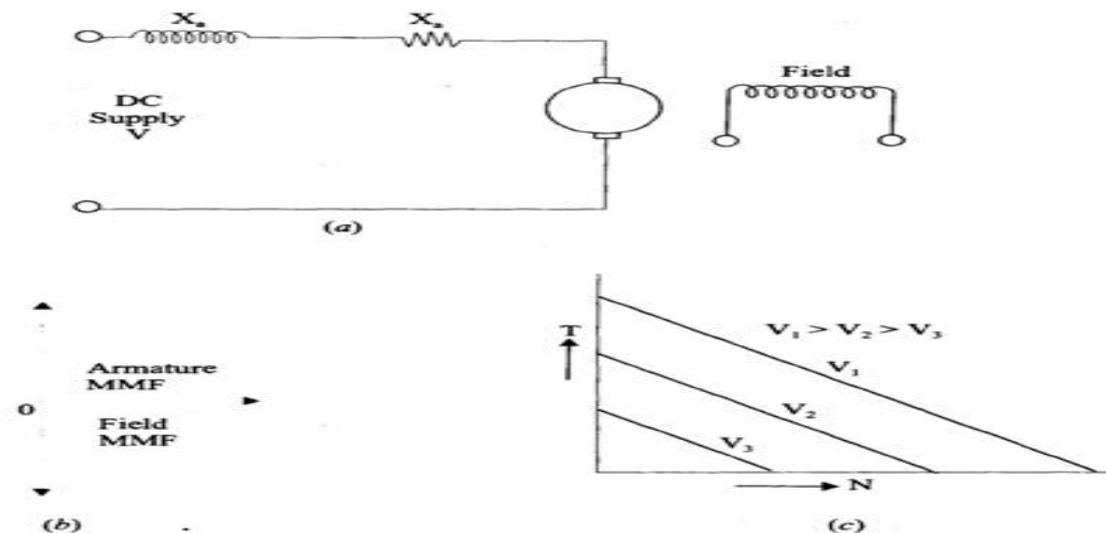
#### Servomotor:

The servo system is the one, in which the output is some mechanical variable such as position, velocity or acceleration. The motors used in the servo systems are called servomotors. These motors are usually coupled to the output shaft for power matching. There are two types of servo motors.

1. DC Servomotors
2. AC Servomotors

**1. DC Servomotors:** - D.C. servomotors are separately excited or permanent magnet d.c. servomotor. The armature of d.c servomotor has a large resistance, therefore torque speed characteristics is linear. The torque speed characteristics

Shows in fig(c) and fig(a) shows the schematic diagram of separately excited d.c. servomotor.



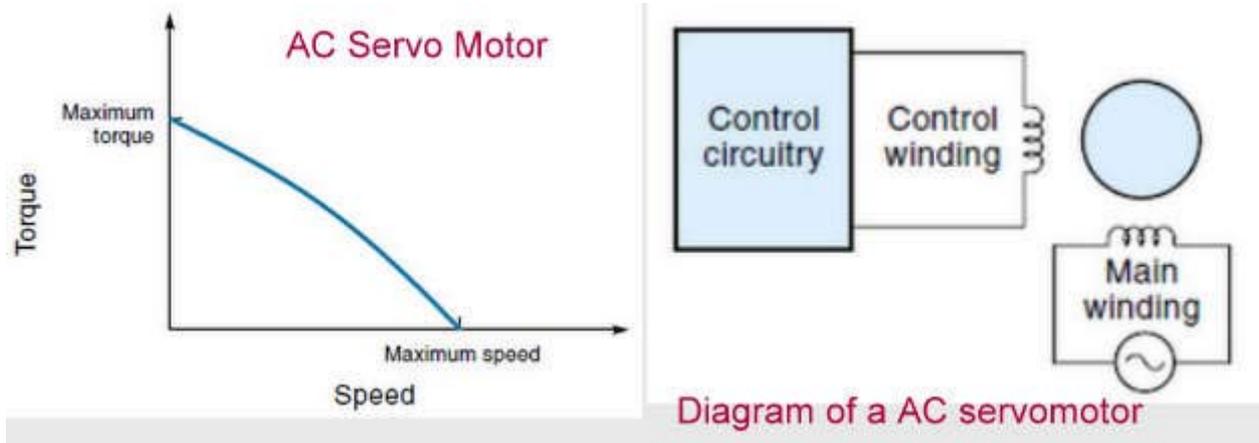
- DC servo motors are controlled by DC command signals applied directly to coils. Time constant for field circuit is large, due to large time constant, the response is slow and therefore they are not commonly used.
- The magnetic fields that are formed interact with permanent magnets and cause the rotating member to turn.
- One type of PM uses a wound armature and brushes like a conventional DC motor, but uses magnets as pole pieces
- Another type uses wound field coils and a permanent magnet rotor.

**2. AC Servomotors:** - These motor having two parts namely stator and rotor. A.C. Servomotors are two phase induction motor. The stator has two distributed windings. These windings are displaced from each other by  $90^\circ$ . One winding is called main winding or reference winding. The reference winding is excited by constant a.c. voltage. Other winding is called control winding, these winding is excited by variable control voltage of the same frequency as the reference winding but having a phase displacement of  $90^\circ$  electrical. The variable control voltage for control winding is obtained from a servo amplifier.

The rotor of a.c. servomotors are of two types (a) squirrel cage rotor (b) drag cup type rotor. The squirrel cage rotor having large length and small diameter, so its resistance is very high. The air gap of squirrel cage is kept small. In drag cup type there are two air gaps.

# Control Systems

For the rotor a cup of nonmagnetic conducting material is used. A stationary iron core is placed between the conducting cup to complete the magnetic circuit. The resistance of drag cup type is high and having high starting torque. Fig(a) shows the schematic diagram of two phase a.c. servomotor and fig(b) shows the two types of rotor.



Fig(a)

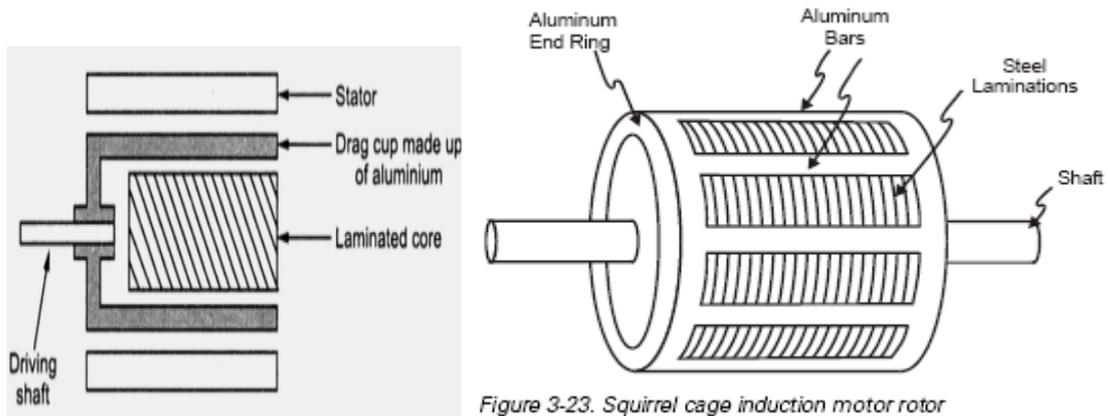
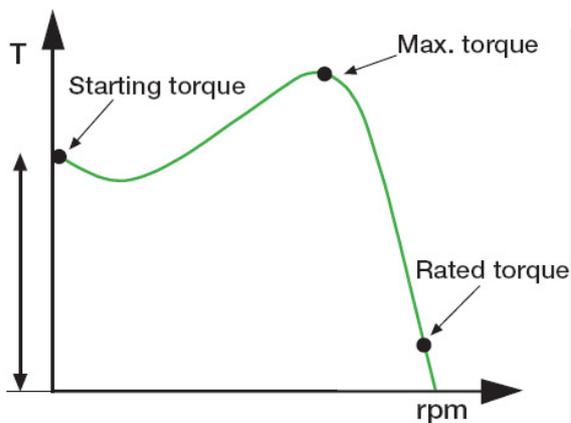


Figure 3-23. Squirrel cage induction motor rotor

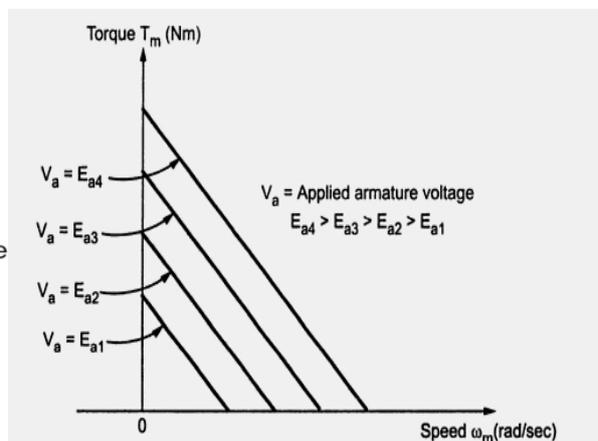
Fig(b)

- Controlled by AC command signals applied to the coils.
- AC Brushless Servo Motor Operates on the same principle as single-phase induction motor.

Torque speed characteristic:



Torque diagram for a typical squirrel cage motor



## Application of servomotors:

Servomotor is widely used in radars, electromechanical actuators, computers, machine tools, tracking and guidance system, process controllers and robots.

## UNIT I

### CONTROL SYSTEM MODELLING

#### 1. What is control system?

A system consists of a number of components connected together to perform a specific function. In a system when the output quantity is controlled by varying the input quantity then the system is called control system.

#### 2. What are the two major types of control system?

The two major types of control system are open loop and closed loop.

#### 3. Define open loop control system.

The control system in which the output quantity has no effect upon the input quantity is called open loop control system. This means that the output is not feedback to the input for correction.

#### 4. Define closed loop control system.

The control system in which the output has an effect upon the input quantity so as to maintain the desired output values are called closed loop control system.

#### 5. What are the components of feedback control system?

The components of feedback control system are plant, feedback path elements, error detector and controller.

#### 6. Define transfer function.

The T.F of a system is defined as the ratio of the Laplace transform of output to Laplace transform of input with zero initial conditions.

#### 7. What are the basic elements used for modeling mechanical translational system.

Mass, spring and dashpot

#### 8. What are the basic elements used for modeling mechanical rotational system?

Moment of inertia  $J$ , dashpot with rotational frictional coefficient  $B$  and torsion spring with stiffness  $K$

#### 9. Name two types of electrical analogous for mechanical system.

The two types of analogies for the mechanical system are - Force voltage and force current analogy.

#### 10. What is block diagram?

A block diagram of a system is a pictorial representation of the functions performed by each component of the system and shows the flow of signals. The basic elements of block diagram arrow block, branch point and summing point.

#### 11. What is the basis for framing the rules of block diagram reduction technique?

The rules for block diagram reduction technique are framed such that any modification made on the diagram does not alter the input output relation.

#### 12. What is a signal flow graph?

A signal flow graph is a diagram that represents a set of simultaneous algebraic equations. By taking L.T the time domain differential equations governing a control system can be transferred to a set of algebraic equations in s-domain.

**13. What is transmittance?**

The transmittance is the gain acquired by the signal when it travels from one node to another node in signal flow graph.

**14. What is sink and source?**

Source is the input node in the signal flow graph and it has only outgoing branches. Sink is an output node in the signal flow graph and it has only incoming branches.

**15. Define non touching loop.**

The loops are said to be non touching if they do not have common nodes.

**16. Write Masons Gain formula.**

Masons Gain formula states that the overall gain of the system is

$$M = \frac{1}{\Delta} \sum_{k=1}^N P_k \Delta_k = \frac{X_{out}}{X_{in}}$$

Where M = gain between  $X_{in}$  and  $X_{out}$

$X_{out}$  = output node variable

$X_{in}$  = input node variable

N = total number of forward paths

$P_k$  = path gain of the k the forward path

$\Delta$  = 1-(sum of loop gains of all individual loop) + (sum of gain product of all possible combinations of two non touching loops) – (sum of gain products of all possible combination of three non touching loops)

**17. Write the analogous electrical elements in force voltage analogy for the elements of mechanical translational system.**

Force	-	voltage e
Velocity	v-	current i
Displacement	x-	charge q
Frictional coefficient	B-	Resistance R
Mass	M-	Inductance L
Stiffness	K-	Inverse of capacitance 1/C

**18. Write the analogous electrical elements in force current analogy for the elements of mechanical translational system.**

Force	-	current i
Velocity	v	voltage v
Displacement	x-	flux
Frictional coefficient	B-	conductance 1/R
Mass	M	capacitance C
Stiffness	K	Inverse of inductance 1/L

**19. Write the force balance equation of ideal mass element.**

$$F = M \frac{d^2x}{dt^2}$$

20. Write the force balance equation of ideal dashpot element.

$$F = B \frac{dx}{dt}$$

21. Write the force balance equation of ideal spring element.

$$F = Kx$$

22. Distinguish between open loop and closed loop system

**Open loop and Closed loop**

- Inaccurate
- Simple and economical
- The changes in output due to external disturbance are not corrected.
- They are generally stable
- Accurate
- Complex and costlier
- The changes in output due to external disturbances are corrected automatically.
- Great efforts are needed to design a stable system.

23. What is servomechanism?

The servomechanism is a feedback control system in which the output is mechanical position (or time derivatives of position velocity and acceleration).

## Module -2

### Lecture-13

#### Time response Analysis: Standard Test Signals.

### 5. Time response Analysis:

#### Introduction

- After deriving a mathematical model of a system, the system performance analysis can be done in various methods.
- In analyzing and designing control systems, a basis of comparison of performance of various control systems should be made. This basis may be set up by specifying particular test input signals and by comparing the responses of various systems to these signals.
- The system stability, system accuracy and complete evaluation are always based on the time response analysis and the corresponding results
- Next important step after a mathematical model of a system is obtained.
- To analyze the system's performance.
- Normally use the standard input signals to identify the characteristics of system's response
  1. Step function
  2. Ramp function
  3. Impulse function
  4. Parabolic function
  5. Sinusoidal function

It is an equation or a plot that describes the behavior of a system and contains much information about it with respect to time response specification as

Over shooting, settling time, peak time, rise time and steady state error. Time response is formed by the transient response and the steady state response.

$$\text{Time response} = \text{Transient response} + \text{Steady state response}$$

**Transient time response** (Natural response) describes the behavior of the system in its first short time until arrives the steady state value and this response will be our study focus. If the input is step function then the output or the response is called step time response and if the input is ramp, the response is called ramp time response ... etc.

#### Classification of Time Response

Transient response

Steady state response

$$y(t) = y_t(t) + y_{ss}(t)$$

#### Transient Response

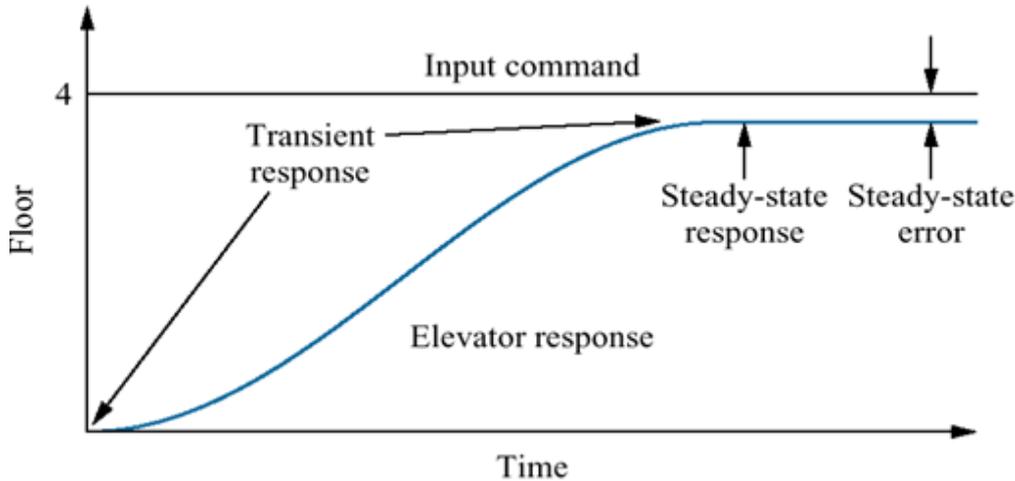
The transient response is defined as the part of the time response that goes to zero as time becomes very large. Thus  $y_t(t)$  has the property

$$\lim_{t \rightarrow \infty} y_t(t) = 0$$

# Control Systems

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The time required to achieve the final value is called transient period. The transient response may be exponential or oscillatory in nature. Output response consists of the sum of forced response (from the input) and natural response (from the nature of the system). The transient response is the change in output response from the beginning of the response to the final state of the response and the steady state response is the output response as time is approaching infinity (or no more changes at the output).

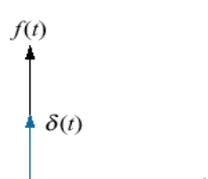
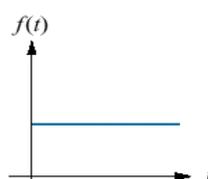
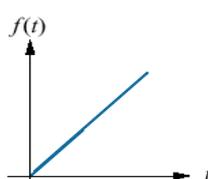
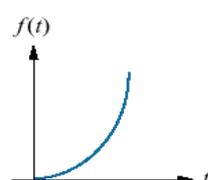
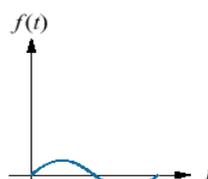


**Steady State Response:** The steady state response is the part of the total response that remains after the transient has died out. For a position control system, the steady state response when compared to with the desired reference position gives an indication of the final accuracy of the system. If the steady state response of the output does not agree with the desired reference exactly, the system is said to have steady state error.

$$\text{Time response} = \text{Transient response} + \text{Steady state response}$$

## Typical Input Signals

1. Impulse Signal
2. Step Signal
3. Ramp Signal
4. Parabolic Signal

Input	Function	Description	Sketch	Use
Impulse	$\delta(t)$	$\delta(t) = \infty$ for $0^- < t < 0^+$ $= 0$ elsewhere $\int_{0^-}^{0^+} \delta(t) dt = 1$		Transient response Modeling
Step	$u(t)$	$u(t) = 1$ for $t > 0$ $= 0$ for $t < 0$		Transient response Steady-state error
Ramp	$tu(t)$	$tu(t) = t$ for $t \geq 0$ $= 0$ elsewhere		Steady-state error
Parabola	$\frac{1}{2}t^2u(t)$	$\frac{1}{2}t^2u(t) = \frac{1}{2}t^2$ for $t \geq 0$ $= 0$ elsewhere		Steady-state error
Sinusoid	$\sin \omega t$			Transient response Modeling Steady-state error

## Time Response Analysis & Design

- Two types of inputs can be applied to a control system
- Command Input or Reference Input  $y_r(t)$
- Disturbance Input  $w(t)$  (External disturbances  $w(t)$  are typically uncontrolled variations in the load on a control system)
- In systems controlling mechanical motions, load disturbances may represent forces.
- In voltage regulating systems, variations in electrical load area major source of disturbances.

## Test Signals

1. Input	$r(t)$	$R(S)$	
2. Step input	$A$	$A/S$	
3. Ramp input	$At$	$A/S^2$	
4. Parabolic input	$At^2/2$	$A/S^3$	
5. Impulse input	$\delta(t)$	$1$	

First-order system time response

- Transient
- Steady-state

### First Order System

$$\frac{Y(s)}{R(s)} = \frac{K}{1 + K + sT} \cong \frac{K}{1 + sT}$$

### Step Response of First Order System

Evolution of the transient response is determined by the pole of the transfer function at  $s = -1/t$  where  $t$  is the time constant. Also, the step response can be found:

$$\left(s + \frac{1}{T}\right) C(s) = \frac{k}{s}$$

$$C(s) = \frac{\frac{k}{T}}{s + \frac{1}{T}} = \frac{k}{s} - \frac{k}{s + \frac{1}{T}}$$

$$C(t) = k\left(1 - e^{-\frac{t}{T}}\right)u(t)$$

Impulse response  $\frac{k}{1 + sT}$  Exponential

Step response  $\frac{k}{s} - \frac{k}{s + \frac{1}{T}}$  Step, Exponential

Ramp response  $\frac{k}{s^2} - \frac{kT}{s} - \frac{kT}{s + \frac{1}{T}}$  Ramp, Step, Exponential

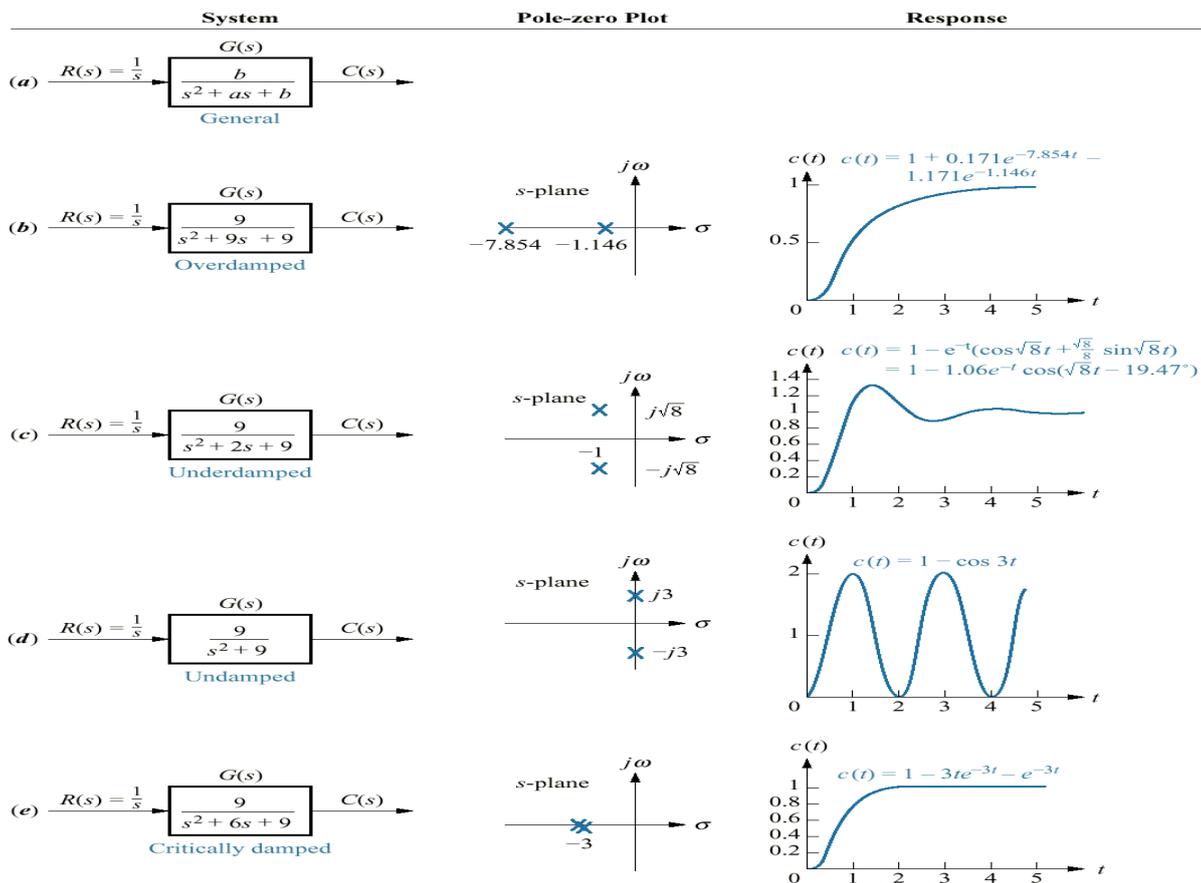
**Second-order systems:** LTI second-order system

$$G(s) = \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$(s^2 + 2\zeta\omega_n s + \omega_n^2)C(s) = \omega_n^2 R(s)$$

$$C(t) + 2\zeta\omega_n C(t) + \omega_n^2 C(t) = \omega_n^2 r(t)$$

Second-Order Systems



### Second order system responses

Over damped response:

Poles: Two real at  $-\sigma_1 \quad -\sigma_2$

Natural response: Two exponentials with time constants equal to the reciprocal of the pole location

$$C(t) = k_1 e^{-\sigma_1 t} + k_2 e^{-\sigma_2 t}$$

### Under damped response:

Poles: two complexes at  $-\sigma_d \pm j\omega_d$

Natural response: Damped sinusoid with an exponential envelope whose time constant is equal to the reciprocal of the pole's radian frequency of the sinusoid, the damped frequency of oscillation, is equal to the imaginary part of the poles

$$C(t) = A e^{-\sigma_d t} \cos(\omega_d t - \Phi)$$

### Un-damped response:

Poles: Two imaginary at  $\pm j\omega_1$

Natural response: Undamped sinusoid with radian frequency equal to the imaginary part of the poles

$$C(t) = A \cos(\omega_1 t - \Phi)$$

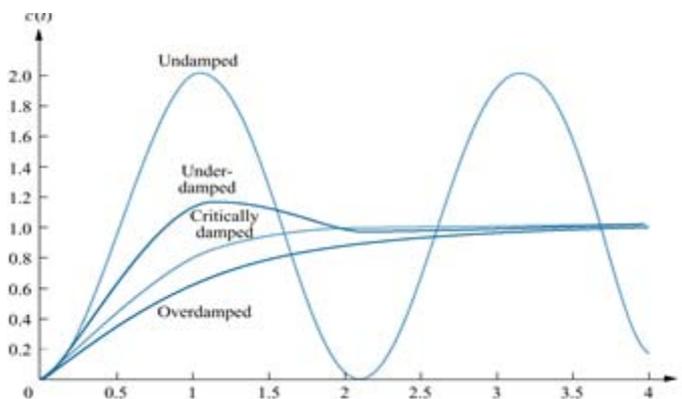
### Critically damped responses:

Poles: Two real at

Natural response: One term is an exponential whose time constant is equal to the reciprocal of the pole location. Another term product of time and an exponential with time constant equal to the reciprocal of the pole location

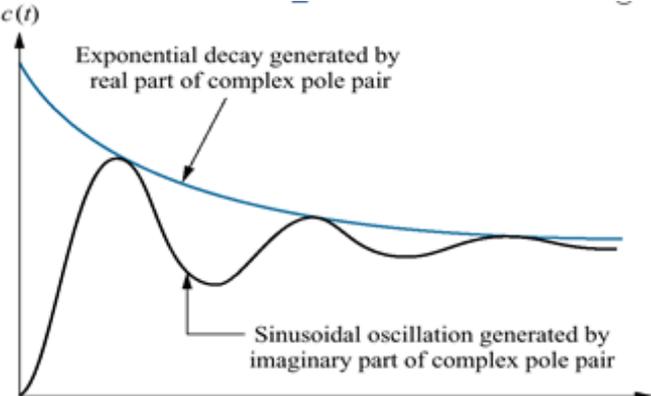
$$C(t) = k_1 e^{-\sigma_1 t} + k_2 t e^{-\sigma_1 t}$$

### Second order system responses damping cases



**Second- order step response**

Complex poles



## Steady State Error:

- Consider a unity feedback system
- Transfer function between  $e(t)$  and  $r(t)$

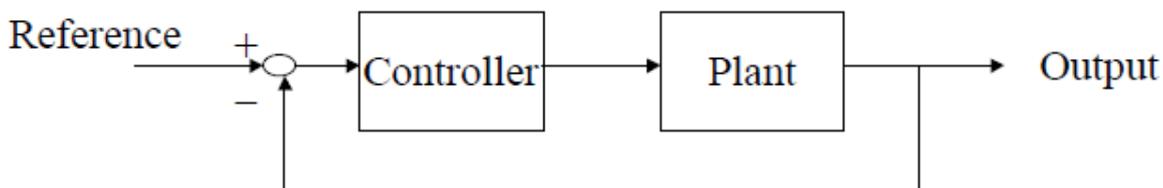
$$\frac{E(s)}{R(s)} = \frac{1}{1 + G(s)} \text{ or } E(s) = \frac{R(s)}{1 + G(s)}$$

Steady state error is

$$e_{ss} = \lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} sE(s) = \lim_{s \rightarrow 0} \frac{sR(s)}{1 + G(s)}$$

Type of system	Error constants			Steady state error $e_{ss}$		
	$K_v$	$K_r$	$K_a$	Unit step input	Unit ramp input	Unit parabolic input
0	K	0	0	$1/(1+K)$	$\infty$	$\infty$
1	$\infty$	K	0	0	$1/K$	$\infty$
2	$\infty$	$\infty$	K	0	0	$1/K$
3	$\infty$	$\infty$	$\infty$	0	0	0

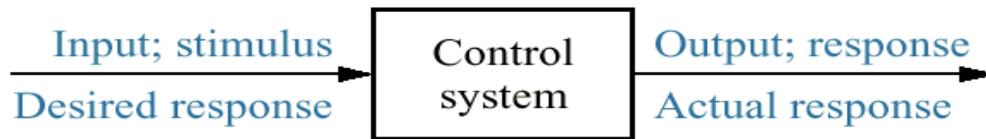
### Output Feedback Control Systems



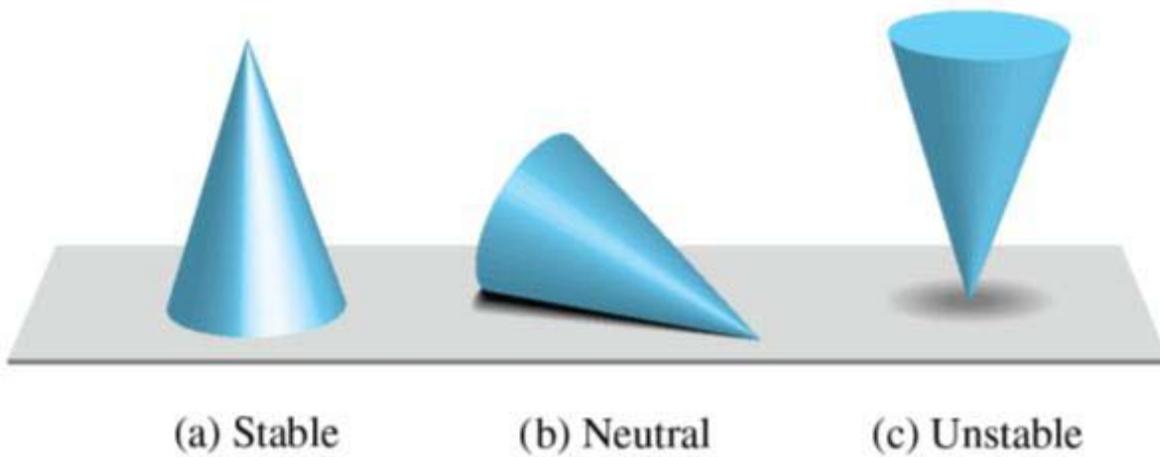
Feedback only the output signal

- Easy access
- Obtainable in practice

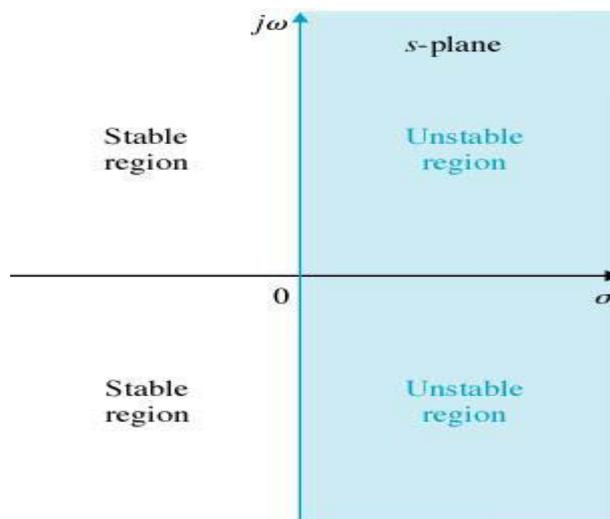
6.Stability Theory: Stability and Algebraic Criteria:- A system is stable if any bounded input produces a bounded output for all bounded initial conditions.



## Basic concept of stability



## Stability of the system and roots of characteristic equations



## Characteristic Equation

# Control Systems

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Consider an  $n$ th-order system whose the characteristic equation (which is also the denominator of the transfer function) is

$$\alpha(s) = \alpha_0 s^n + \alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \dots + \alpha_{n-1} s^1 + \alpha_n s^0$$

## Routh Hurwitz Criterion

- Goal: Determining whether the system is stable or unstable from a characteristic equation in polynomial form without actually solving for the roots
- Routh's stability criterion is useful for determining the ranges of coefficients of polynomials for stability, especially when the coefficients are in symbolic (non numerical) form
- To find  $K_{mar}$  &  $\omega$

### A necessary condition for Routh's Stability

- A necessary condition for stability of the system is that all of the roots of its characteristic equation have negative real parts, which in turn requires that all the coefficients be positive.
- A necessary (but not sufficient) condition for stability is that all the coefficients of the polynomial characteristic equation are positive & none of the co-efficient vanishes.
- Routh's formulation requires the computation of a triangular array that is a function of the coefficients of the polynomial characteristic equation.
- A system is stable if and only if *all* the elements of the first column of the Routh array are positive

### Method for determining the Routh array

Consider the characteristic equation

$$a(s) = a_0s^n + a_1s^{n-1} + a_2s^{n-2} + \dots + a_{n-1}s^1 + a_ns^0$$

### Routh array method

Then add subsequent rows to complete the Routh array

Compute elements for the 3rd row:

$$b_1 = \frac{a_1a_2 - a_0a_3}{a_1}$$

$$b_2 = \frac{a_1a_4 - a_0a_5}{a_1}$$

$$b_3 = \frac{a_1a_6 - a_0a_7}{a_1}$$

...

Given the characteristic equation,

$$a(s) = s^6 + 4s^5 + 3s^4 + 2s^3 + s^2 + 4s + 4$$

Is the system described by this characteristic equation stable?

Answer:

- All the coefficients are positive and nonzero
- Therefore, the system satisfies the necessary condition for stability
- We should determine whether any of the coefficients of the first column of the Routh array are negative

$$S^6: 1 \quad 3 \quad 1 \quad 4$$

$$S^5: 4 \quad 2 \quad 4 \quad 0$$

$$S^4: 2.5 \quad 0 \quad 4$$

$$S^3: 2 \quad -2.4 \quad 0$$

$$S^2: 3 \quad 4$$

$$S^1: -1.6 \quad 0$$

$$S^0: 4$$

The elements of the 1<sup>st</sup> column are not all positive. Then the system is unstable.

## Special cases of Routh's criteria

Case 1: All the elements of a row in a RA are zero

- Form Auxiliary equation by using the co-efficient of the row which is just above the row of zeros
- Find derivative of the A.E.
- Replace the row of zeros by the co-efficient of  $dA(s)/ds$
- complete the array in terms of these coefficients
- analyze for any sign change, if so, unstable
- no sign change, find the nature of roots of AE
- non-repeated imaginary roots - marginally stable
- repeated imaginary roots – unstable

Case 2: First element of any of the rows of RA is

## Control Systems

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- Zero and the same remaining row contains at least one non-zero element
- Substitute a small positive no. " $\epsilon$ " in place of zero and complete the array.
- Examine the sign change by taking  $\text{Lt } \epsilon = 0$

#### 7. Root locus Technique:

- Introduced by W. R. Evans in 1948
- Graphical method, in which movement of poles in the s-plane is sketched when some parameter is varied
- The path taken by the roots of the characteristic equation when open loop gain K is varied from 0 to  $\infty$  are called root loci
- Direct Root Locus =  $0 < k < \infty$
- Inverse Root Locus =  $-\infty < k < 0$

#### Root Locus Analysis:

- The roots of the closed-loop characteristic equation define the system characteristic responses
- Their location in the complex s-plane lead to prediction of the characteristics of the time domain responses in terms of:
  - o damping ratio,  $\zeta$
  - o natural frequency,  $\omega_n$
  - o damping constant,  $\sigma \rightarrow$  first-order modes
  - o Consider how these roots change as the loop gain is varied from 0 to  $\infty$

#### Basics of Root Locus:

- Symmetrical about real axis
- RL branch starts from OL poles and terminates at OL zeroes
- No. of RL branches = No. of poles of OLTF
- Centroid is common intersection point of all the asymptotes on the real axis
- Asymptotes are straight lines which are parallel to RL going to  $\infty$  and meet the RL at  $\infty$
- No. of asymptotes = No. of branches going to  $\infty$
- At Break Away point, the RL breaks from real axis to enter into the complex plane
- At BI point, the RL enters the real axis from the complex plane

#### Constructing Root Locus:

- Locate the OL poles & zeros in the plot
- Find the branches on the real axis
- Find angle of asymptotes & centroid
- $\Phi_a = \pm 180^\circ(2q+1) / (n-m)$
- $\sigma_a = (\sum \text{poles} - \sum \text{zeroes}) / (n-m)$
- Find BA and BI points
- Find Angle Of departure (AOD) and Angle Of Arrival (AOA)
- $\text{AOD} = 180^\circ - (\text{sum of angles of vectors to the complex pole from all other poles}) + (\text{Sum of angles of vectors to the complex pole from all zero})$

# Control Systems

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- AOA = 180°- (sum of angles of vectors to the complex zero from all other zeros) + (sum of angles of vectors to the complex zero from poles)
- Find the point of intersection of RL with the imaginary axis.

## Application of the Root Locus Procedure:

Step 1: Write the characteristic equation as

$$1 + F(s) = 0$$

Step 2: Rewrite preceding equation into the form of poles and zeros as follows

$$1 + K \frac{\prod_{j=1}^m (s - z_j)}{\prod_{i=1}^n (s - p_i)} = 0$$

Step 3:

- Locate the poles and zeros with specific symbols, the root locus begins at the open-loop poles and ends at the open loop zeros as K increases from 0 to infinity
- If open-loop system has n-m zeros at infinity, there will be n-m branches of the root locus approaching the n-m zeros at infinity

Step 4:

- The root locus on the real axis lies in a section of the real axis to the left of an odd number of real poles and zeros

Step 5:

- The number of separate loci is equal to the number of open-loop poles

Step 6:

- The root loci must be continuous and symmetrical with respect to the horizontal real axis

Step 7:

- The loci proceed to zeros at infinity along asymptotes centered at centroid and with angles

$$\sigma_a = \frac{\sum_{i=1}^n p_i - \sum_{j=1}^m z_j}{n - m}$$

$$\phi_a = \frac{(2k + 1)\pi}{n - m} \quad (k = 0, 1, 2, \dots, n - m - 1)$$

Step 8:

- The actual point at which the root locus crosses the imaginary axis is readily evaluated by using Routh's criterion

Step 9:

- Determine the breakaway point d (usually on the real axis)

Step 10:

- Plot the root locus that satisfy the phase criterion

$$\angle P(s) = (2k + 1)\pi \quad k = 1, 2, \dots$$

Step 11:

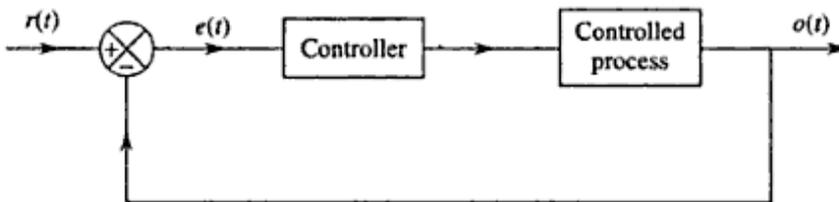
- Determine the parameter value K1 at a specific root using the magnitude criterion

$$K_1 = \frac{\prod_{i=1}^n |(s - p_i)|}{\prod_{j=1}^m |(s - z_j)|} \Big|_{s=s_1}$$

## 8. Compensation Technique:

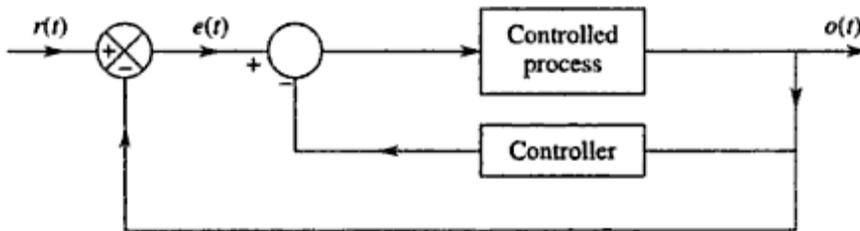
Series Compensation or Cascade Compensation

- This is the most commonly used system where the controller is placed in series with the controlled process.
- Figure shows the series compensation



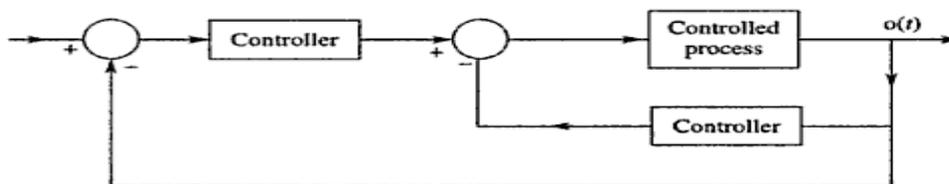
*Series compensation.*

Feedback Compensation or Parallel Compensation



*Feedback compensation or parallel compensation.*

Series-Feedback Compensation



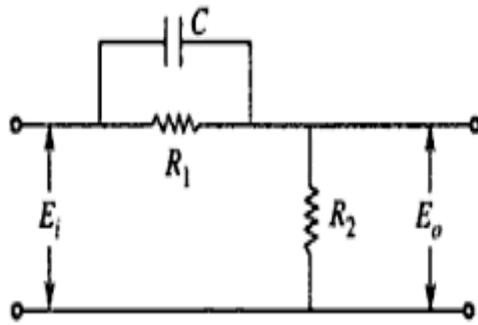
*Series-feedback compensation.*

### Lead Compensator

It has a zero and a pole with zero closer to the origin. The general form of the transfer function of the lead compensator is

$$G(s) = \frac{s + \frac{1}{\tau}}{s + \frac{1}{\beta\tau}}$$

$$G(j\omega) = \beta \frac{(\tau j\omega + 1)}{\beta\tau j\omega + 1}$$



Lead compensator.

Here,

$$\bar{E}_o(s) = \frac{E_i(s)R_2}{R_1 \frac{1}{Cs} + R_2} \cdot \frac{R_2}{R_1 + \frac{1}{Cs}}$$

$$\frac{E_o(s)}{E_i(s)} = \frac{R_2}{R_1 \frac{1}{Cs} + R_2 \left( R_1 + \frac{1}{Cs} \right)} = \frac{R_2 R_1 + \frac{R_2}{Cs}}{R_1 R_2 + \frac{1}{Cs} (R_1 + R_2)}$$

$$= \frac{Cs R_1 R_2 + R_2}{Cs R_1 R_2 + R_1 + R_2} = \frac{R_2 (Cs R_1 + 1)}{(R_1 + R_2) \left( \frac{Cs R_1 R_2}{R_1 + R_2} + 1 \right)}$$

$$= R_2 / \left( (R_1 + R_2) \left( \frac{Cs R_1 R_2}{R_1 + R_2} + 1 \right) \right)$$

Substituting

$$\tau = CR_1; \beta\tau = \frac{CR_1 R_2}{R_1 + R_2} \quad (\because \tau = CR_1)$$

Transfer function

$$G(s) = \beta \frac{\tau s + 1}{\beta\tau s + 1}$$

### Lag Compensator

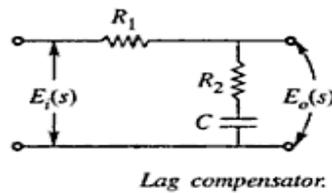
It has a zero and a pole with the zero situated on the left of the pole on the negative real axis. The general form of the transfer function of the lag compensator is

$$G(s) = \left( \frac{s + \frac{1}{\tau}}{s + \frac{1}{\alpha\tau}} \right) = \frac{\alpha(\tau s + 1)}{\alpha\tau s + 1} \quad \text{where } \alpha > 1, \tau > 0$$

Therefore, the frequency response of the above transfer function will be

$$G(s) = \frac{\alpha(\tau j\omega + 1)}{\alpha\tau j\omega + 1}$$

$$\bar{E}_o(s) = \frac{E_i(s)}{R_1 + R_2 + \frac{1}{Cs}} \left( R_2 + \frac{1}{sC} \right)$$



$$\begin{aligned} \frac{E_o(s)}{E_i(s)} &= \frac{\left(R_2 + \frac{1}{sC}\right)}{R_1 + R_2 + \frac{1}{Cs}} \\ &= \frac{(R_2Cs + 1)}{[(R_1 + R_2)Cs + 1]} \\ &= \frac{R_2C \left(s + \frac{1}{R_2C}\right)}{(R_1 + R_2)C \left(s + \frac{1}{(R_1 + R_2)C}\right)} \\ &= \frac{R_2 \left(s + \frac{1}{R_2C}\right)}{[(R_1 + R_2) \left(s + \frac{1}{(R_1 + R_2)C}\right)]} \\ &= \frac{R_2 \left(s + \frac{1}{R_2C}\right)}{[(R_1 + R_2) \left(s + \frac{R_2}{(R_1 + R_2)R_2C}\right)]} \end{aligned}$$

Now comparing with

$$\begin{aligned} G(s) &= \left(\frac{s + \frac{1}{\tau}}{s + \frac{1}{\alpha\tau}}\right) \\ \frac{1}{\tau} &= \frac{1}{R_2C} \quad \frac{1}{\alpha\tau} = \frac{R_2}{(R_1 + R_2)R_2C} \\ \frac{1}{\alpha\tau} &= \frac{R_2}{(R_1 + R_2)\tau} \quad \left(\text{as } \frac{1}{\tau} = \frac{1}{R_2C}\right) \\ \alpha &= \frac{R_1 + R_2}{R_2} \end{aligned}$$

Therefore

$$\frac{E_o(s)}{E_i(s)} = \frac{1}{\alpha} \left(\frac{s + \frac{1}{\tau}}{s + \frac{1}{\alpha\tau}}\right)$$

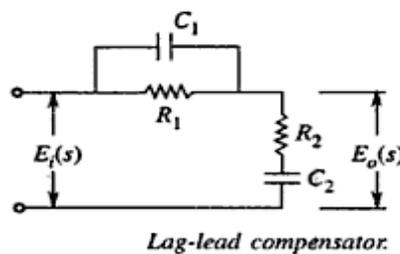
## Lag-Lead Compensator

The lag-lead compensator is the combination of a lag compensator and a lead compensator. The lag section is provided with one real pole and one real zero, the pole being to the right of zero, whereas the lead section has one real pole and one real zero with the zero being to the right of the pole.

The transfer function of the lag-lead compensator will be

$$G(s) = \left( \frac{s + \frac{1}{\tau_1}}{s + \frac{1}{\alpha\tau_1}} \right) \left( \frac{s + \frac{1}{\tau_2}}{s + \frac{1}{\beta\tau_2}} \right)$$

The figure shows lag lead compensator



Where  $\alpha > 1$ ,  $\beta > 1$

$$E_o(s) = \frac{E_i(s)}{\frac{R_1 \frac{1}{sC_1}}{R_1 + \frac{1}{sC_1}} + R_2 + \frac{1}{sC_2}} \left( R_2 + \frac{1}{sC_2} \right)$$

$$\frac{E_o(s)}{E_i(s)} = \frac{\left( R_2 + \frac{1}{sC_2} \right) \left( R_2 + \frac{1}{sC_2} \right)}{R_1 \frac{1}{sC_1} + \left( R_2 + \frac{1}{sC_2} \right) \left( R_1 + \frac{1}{sC_1} \right)}$$

$$= \frac{\frac{R_1}{sC_1} + \frac{[(R)_2 sC_2 + 1][(R)_1 sC_1 + 1]}{sC_2}}{\frac{(sC_1 R_1 + 1)(sC_2 R_2 + 1)}{s^2 C_1 C_2}}$$

$$= \frac{[(R_1 sC_2 + R)_2 sC_2 + 1 + R_1 R_2 s^2 C_1 C_2 + R_1 sC_1]}{s^2 C_1 C_2}$$

$$= \frac{(sC_1 R_1 + 1)(sC_2 R_2 + 1)}{s^2 R_1 R_2 C_1 C_2 + [s(R)_1 C_1 + R_2 C_2] + 1 + R_1 s C_2}$$

$$= \frac{C_1 R_1 C_2 R_2 \left( s + \frac{1}{C_1 R_1} \right) \left( s + \frac{1}{C_2 R_2} \right)}{R_1 R_2 C_1 C_2 \left[ s^2 + s \left\{ \frac{1}{C_1 R_1} + \frac{1}{C_2 R_2} + \frac{1}{C_1 R_2} \right\} + \frac{1}{C_1 R_1 C_2 R_2} \right]}$$

$$= \frac{\left( s + \frac{1}{C_1 R_1} \right) \left( s + \frac{1}{C_2 R_2} \right)}{\left[ s^2 + s \left\{ \frac{1}{C_1 R_1} + \frac{1}{C_2 R_2} + \frac{1}{C_1 R_2} \right\} + \frac{1}{C_1 R_1 C_2 R_2} \right]}$$

The above transfer functions are comparing with

$$G(s) = \frac{\left(s + \frac{1}{\tau_1}\right)\left(s + \frac{1}{\tau_2}\right)}{\left(s + \frac{1}{\alpha\tau_1}\right)\left(s + \frac{1}{\beta\tau_2}\right)}$$

Then

$$\frac{1}{\tau_1} = \frac{1}{C_1 R_1}, \quad \frac{1}{\tau_2} = \frac{1}{C_2 R_2}$$

$$\frac{1}{\alpha\tau_1} + \frac{1}{\beta\tau_2} = \frac{1}{C_1 R_1} + \frac{1}{C_2 R_2} + \frac{1}{C_1 R_2}$$

$$\frac{1}{\alpha\beta\tau_1\tau_2} = \frac{1}{C_1 C_2 R_1 R_2}$$

$$\tau_1 = C_1 R_1$$

$$\tau_2 = C_2 R_2$$

$$\alpha\beta\tau_1\tau_2 = C_1 C_2 R_1 R_2$$

$$\alpha\beta = 1 \text{ or } \beta = \frac{1}{\alpha}$$

Therefore

$$G(s) = \left(\frac{s + \frac{1}{\tau_1}}{s + \frac{1}{\alpha\tau_1}}\right)\left(\frac{s + \frac{1}{\tau_2}}{s + \frac{1}{\tau_2}}\right)$$

Where  $\alpha > 1$ ,

$$\frac{1}{R_1 C_1} + \frac{1}{R_2 C_1} + \frac{1}{R_2 C_2} = \frac{1}{\alpha\tau_1} + \frac{1}{\tau_2}$$

# Module -3

Lecture-28, 29

Frequency Response Analysis: Frequency domain specifications

## What is Frequency Response?

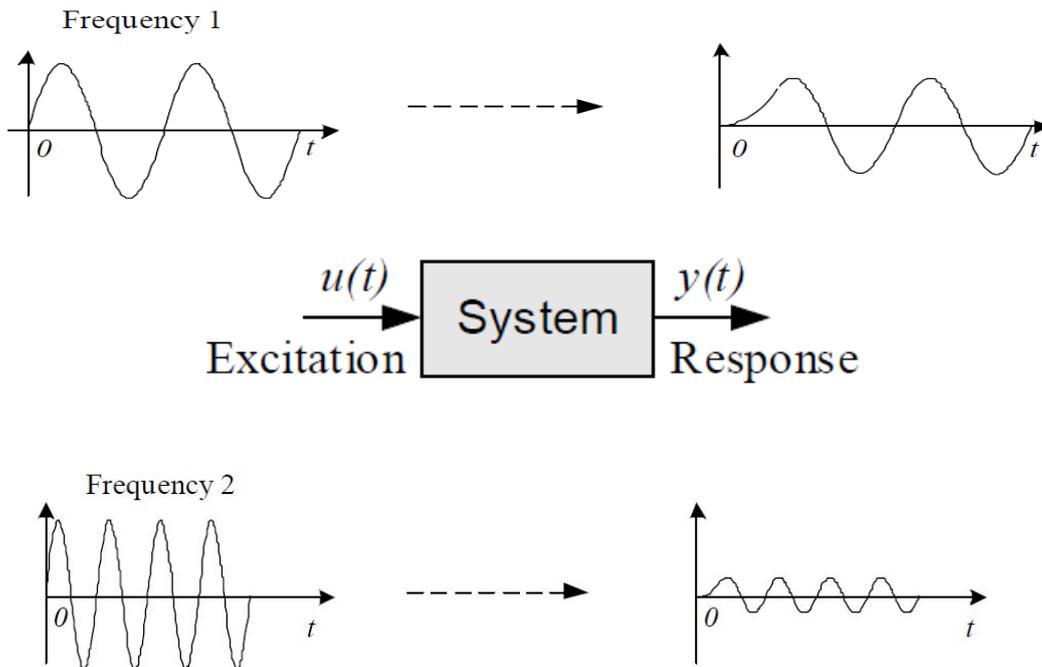
Consider a system with a sinusoidal input

$$r(t) = A \sin \omega t$$

Under steady state, the system output as well as signals at all other points in the system are sinusoidal. The steady state output may be written as

$$c(t) = B \sin (\omega t + \phi)$$

The magnitude and phase relationship between the sinusoidal input and the steady state output of a system is termed as frequency response. In linear time-invariant systems, the frequency response is independent of the amplitude and phase of the input signal.



## Advantages of Frequency Response Analysis:

The frequency response test on a system or a component is normally performed by keeping the amplitude  $A$  fixed and determining  $B$  and  $\phi$  for a suitable range of frequencies. Signal generators and precise measuring instruments are readily available for various ranges of frequencies and amplitudes.

- Whenever it is not possible to obtain the form of the transfer function of a system through analytical techniques, the necessary information to compute the transfer function can be extracted by performing the frequency response test on the system.
- The design and parameter adjustment of the open loop transfer function of a system for specified closed loop performance is carried out somewhat more easily in frequency domain than in time domain.
- The effect of noise disturbance and parameter variations are relatively easy to visualize and assess through frequency response.
- The Nyquist Stability criterion is a powerful frequency domain method of extracting the information regarding stability as well as relative stability of a system without the need to find roots of the characteristic equation.

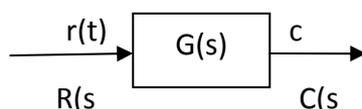
## How to obtain Steady-State Outputs to Sinusoidal Inputs?

The Laplace Transform of the output of a linear single-input, single-output system with transfer function  $G(s)$  can be expressed in terms of the input as

$$C(s) = G(s) \cdot R(s)$$

We know, in general that  $s = \sigma + j\omega$ . However, it will be shown here that, for sinusoidal steady-state analysis, we shall replace  $s$  by its imaginary component  $j\omega$  only, since in steady state, the contribution of the real part  $\sigma$  will disappear for a stable system.

Consider the stable, linear system shown below.



Let us assume that the input signal  $r(t) = A \sin \omega t$ .

Suppose that the transfer function  $G(s)$  of the system can be written as a ratio of two polynomials in  $s$  as

$$G(s) = \frac{p(s)}{q(s)} = \frac{p(s)}{(s + a)(s + b)(s + c)}$$

The Laplace Transform of the output of the system is then  $G(s)R(s) = \frac{K(s)}{q(s)}R(s)$ , where  $R(s)$  is the Laplace Transform of the input  $r(t)$ .

$$\begin{aligned} C(s) &= G(s)R(s) = G(s) \frac{A\omega}{s^2 + \omega^2} \\ &= \frac{K_1}{s + a} + \frac{K_2}{s + b} + \frac{K_3}{s + c} + \frac{K}{s + j\omega} + \frac{K^*}{s + j\omega} \end{aligned}$$

Where,  $K^*$  is the conjugate of  $K$ . The inverse Laplace of the above equation yields

$$c(t) = K_1 e^{-at} + K_2 e^{-bt} + K_3 e^{-ct} + K e^{-j\omega t} + K^* e^{j\omega t}$$

For a stable system, a, b, c have positive real parts. Hence, as  $t$  approaches  $\infty$  at steady state, all the terms in the expression for  $c(t)$  will vanish except the last two terms. Thus at steady state, the response becomes

$$c_{ss}(t) = K e^{-j\omega t} + K^* e^{j\omega t}$$

Regardless of whether there are simple or multiple poles of  $G(s)$ , the contribution due to them to the steady state response will zero.

Where the constant  $K$  can be evaluated as follows:

$$K = G(s) \frac{A\omega}{(s^2 + \omega^2)(s + j\omega)} \Big|_{s=-j\omega} = -\frac{AG(-j\omega)}{2j}$$

$$K^* = G(s) \frac{A\omega}{(s^2 + \omega^2)(s + j\omega)} \Big|_{s=j\omega} = \frac{AG(j\omega)}{2j}$$

Since  $G(j\omega)$  is a complex quantity, it can be written in the form  $G(j\omega) = |G(j\omega)| e^{j\theta(\omega)} = M e^{j\theta(\omega)}$ ,

Where  $M = |G(j\omega)|$  represents the magnitude and  $\theta$  represents the angle of  $G(j\omega)$ .

$$\theta = \angle G(j\omega) = \tan^{-1} \left[ \frac{\text{Imaginary part of } G(j\omega)}{\text{Real part of } G(j\omega)} \right]$$

Similarly,  $G(-j\omega) = |G(-j\omega)| e^{-j\theta(\omega)} = |G(j\omega)| e^{-j\theta(\omega)}$ .

We can now write,

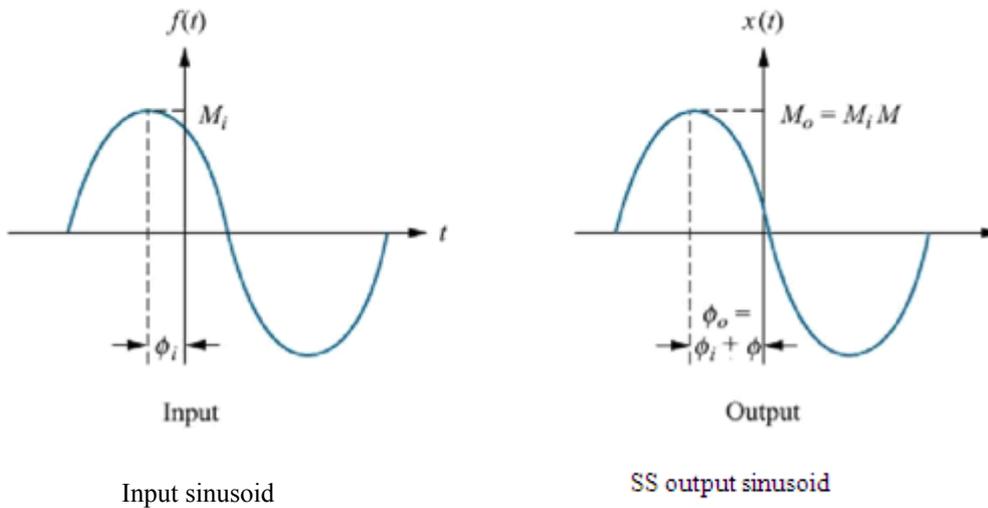
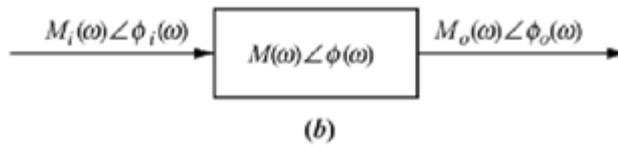
$$c_{ss}(t) = A |G(j\omega)| \frac{e^{j(\omega t + \theta)} - e^{-j(\omega t + \theta)}}{2j}$$

$$= A |G(j\omega)| \sin(\omega t + \theta)$$

$$= B \sin(\omega t + \theta)$$

Where  $B = A |G(j\omega)| = AM$

Hence, for a stable, LTI system, subjected to sinusoidal input, the amplitude of the output is given by the product of that of the input and  $|G(j\omega)|$ , while the phase angle differs from that of the input by an amount  $\theta = \angle G(j\omega)$ . A positive phase angle  $\theta$  is called phase lead where as a negative phase angle is called phase lag.



$$G(s) = \frac{K(s + z_1)(s + z_2) \dots (s + z_k)}{s^m(s + p_1)(s + p_2) \dots (s + p_n)}$$

**Frequency Domain parameters of prototype 2<sup>nd</sup> order systems:**

The closed loop transfer function of a prototype 2<sup>nd</sup> order system is given by

Where  $\zeta$  is the damping factor and  $\omega_n$  is the undamped natural frequency.

The sinusoidal transfer function of the system is obtained by substituting  $s = j\omega$ .

Hence,

$$= \frac{1}{(1 - u^2) + j2\zeta u}$$

Where,  $u = \omega/\omega_n$  is the normalized driving signal frequency

$$M = |T(j\omega)| = 1 / \sqrt{[(1 - u^2)^2 + (2\zeta u)^2]}$$

$$\angle T(j\omega) = \phi = - \tan^{-1} \left[ \frac{2\zeta u}{(1 - u^2)} \right]$$

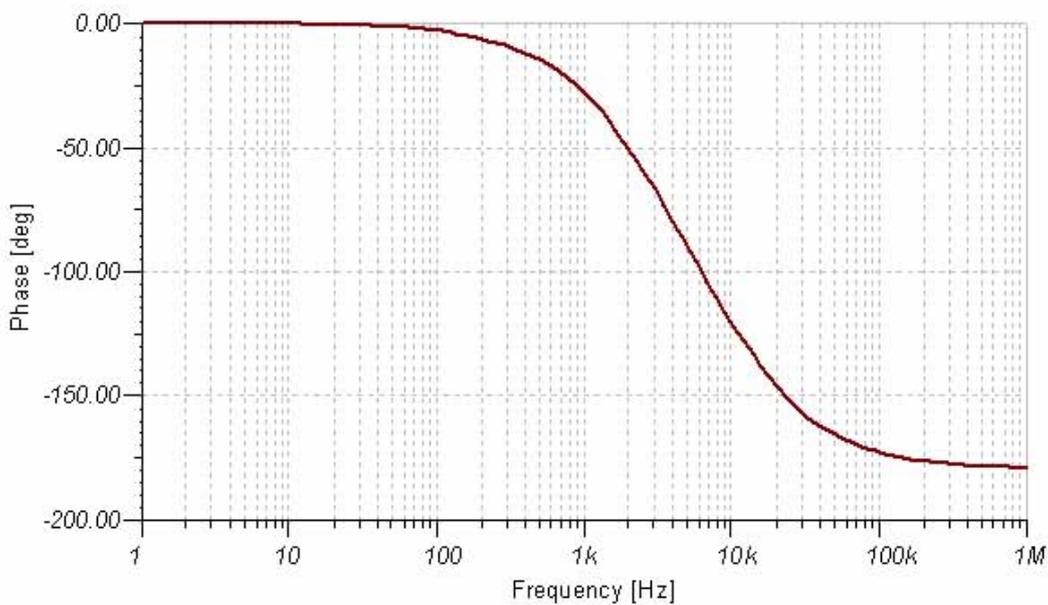
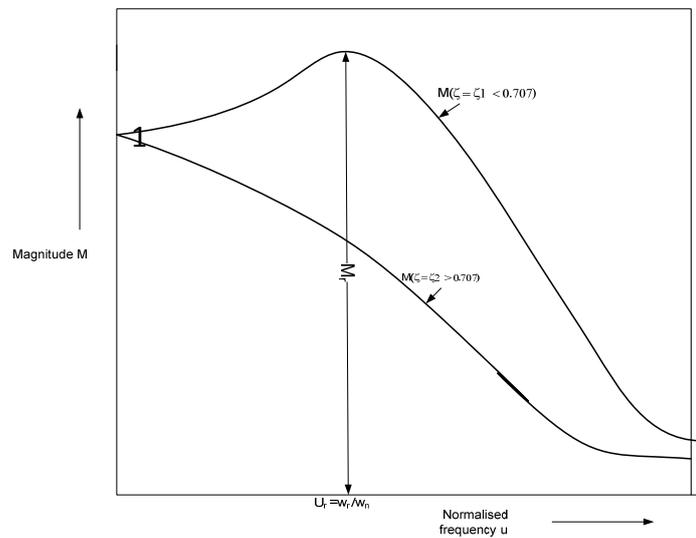
It is seen that when

$$u = 0, M = 1 \text{ and } \phi = 0$$

$$u = 1, M = 1/2\zeta \text{ and } \phi = -\pi/2$$

$$u \rightarrow \infty, M \rightarrow 0 \text{ and } \phi \rightarrow -\pi$$

The Magnitude and Phase angle characteristics for normalized frequency  $u$  for certain values of  $\zeta$  are shown below.



The frequency where  $M$  has a peak value is known as Resonant Frequency. At this frequency, slope of the Magnitude curve is zero. Let  $\omega_r$  be the resonant frequency and  $(u_r = \omega_r/\omega_n)$  be the normalized resonant frequency. Then

$$\begin{aligned} \left(\frac{dM}{du}\right)_{(u = u_r)} &= -\frac{(1)}{2} \frac{(-4(1 - [u_r]^2) u_r + 8(2 u_r))}{[(1 - [u_r]^2)^2 + (2 \zeta u_r)^2]^{3/2}} \\ (4u_r^3 - 4u_r + 8 \zeta u_r) &= 0 \\ u_r &= \sqrt{1 - 2(\zeta^2)} \end{aligned}$$

$$\text{or, } \omega_r = \omega_n \sqrt{1 - 2\xi^2} \quad (1)$$

The maximum value of the magnitude, known as the Resonant Peak is given by

$$(2)$$

The phase angle  $\phi$  of  $T(j\omega)$  at the resonant frequency is given by

$$\sqrt{\left(\frac{\sqrt{1 - 2\xi^2}}{2\xi}\right)}$$

From Eqn. (1) and Eqn. (2), it is seen that as  $\xi$  approaches zero,  $\omega_r$  approaches  $\omega_n$  and  $M_r$  approaches infinity. For  $0 < \xi \leq 1/\sqrt{2}$ , the resonant frequency always has a value less than  $\omega_n$  and the resonant peak has a value greater than 1.

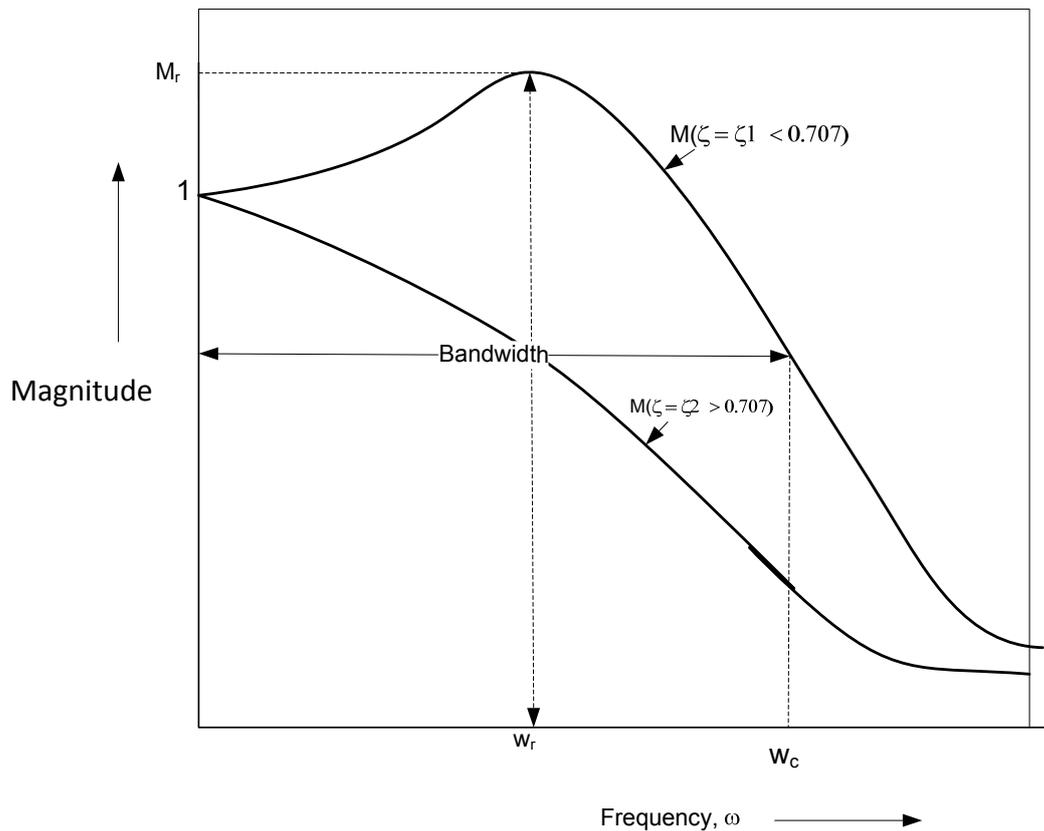
For  $\xi > 1/\sqrt{2}$ , it is seen that  $\frac{dM}{d\omega}$ , slope of the magnitude curve does not become zero for any real value of  $\omega$ . For this range of  $\xi$ , the magnitude of  $M$  decreases monotonically from  $M=1$  at  $\omega=0$  with increasing  $\omega$ , as shown in the above figure. It therefore follows that for  $\xi > 1/\sqrt{2}$ , there is no resonant peak and as such the greatest value of  $M$  equals 1.

As is evident from the above equations, for a second order system, the resonant peak  $M_r$  of its frequency response is indicative of its damping factor  $\xi$  for  $0 < \xi \leq 1/\sqrt{2}$ , and the resonant frequency  $\omega_r$  of the frequency response is indicative of its natural frequency for a given  $\xi$  and hence indicative of its speed of response (as  $t_{1\%} = 4 / (\zeta \omega_n)$ ).  $M_r$  and  $\omega_r$  of the frequency response could thus be used as performance indices for a second order system.

For  $\omega \gg \omega_n$ ,  $M$  decreases monotonically. The frequency at which  $M$  has a value  $1/\sqrt{2}$  is of special significance and is called the cut-off frequency  $\omega_c$ . The signal frequencies above the cut-off frequency are greatly attenuated in passing through a system.

For feedback control systems, the range of frequencies over which  $M$  is equal to or greater than  $1/\sqrt{2}$  is defined as the bandwidth  $\omega_b$ . Control systems being low-pass filters (At zero frequency,  $M=1$ ), the bandwidth  $\omega_b$  is equal to the cut-off frequency  $\omega_c$ .

In general, the bandwidth of a control system indicates the noise-filtering characteristic of the system. Also, the bandwidth gives a measure of the transient response properties as observed below.



The normalized bandwidth  $\omega_b = \omega_b / \omega_n$  of the second order system under consideration can be determined as follows:

$$M = \frac{1}{\sqrt{[(1 - \omega_b^2)^2 + (2\zeta\omega_b)^2]}} = \frac{1}{\sqrt{2}}$$

$$\omega_b^4 - 2(1 - 2\zeta^2) \omega_b^2 + 1 = 0$$

Solving for  $\omega_b$ , we get,

$$\omega_b = [1 - 2\zeta^2 + \sqrt{(2\zeta^2 - 4\zeta^2 + 4\zeta^4)}]^{1/2}$$

It can be approximated in linear form as

$$\omega_b = -1.19\zeta + 1.85$$

We thus observe that the normalized bandwidth is a function of damping only. The de-normalized bandwidth can be written as

$$\omega_b = \omega_n [1 - 2\zeta^2 + \sqrt{(2\zeta^2 - 4\zeta^2 + 4\zeta^4)}]^{1/2}$$

### Correlation between Time Domain and Frequency Domain:

Let us consider the step response of the second order system. The peak overshoot  $M_p$  of the step response for  $0 < \xi \leq 1$  is

$$M_{1p} = s^{-1}(-\pi / (1 - (\zeta^2)))$$

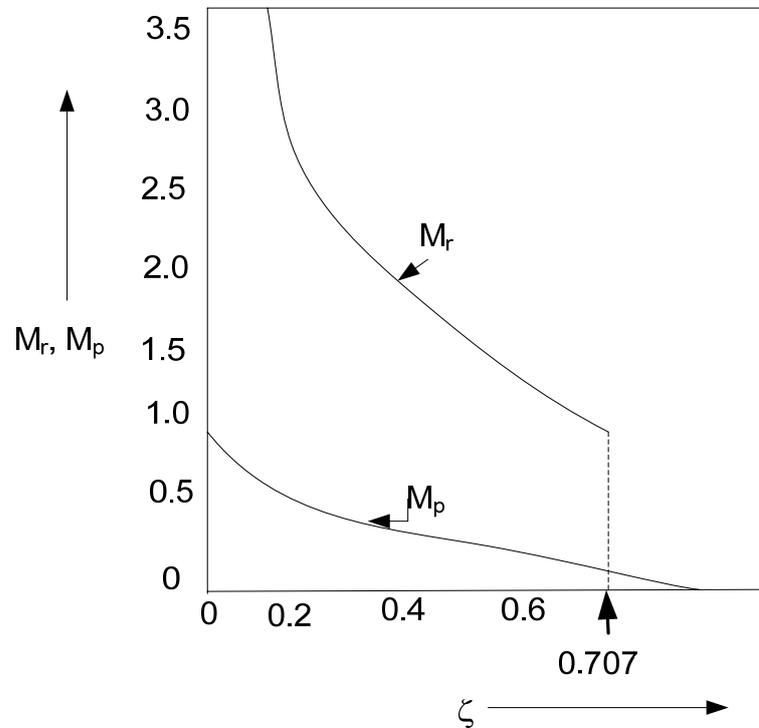
The comparison of  $M_r$  and  $M_p$  plots is shown below. It shows that for  $0 < \zeta \leq 1/\sqrt{2}$ , the two performance indices are correlated as both are functions of the system damping factor  $\zeta$  only. It means that a system with a given value of  $M_r$  of its frequency response, must exhibit a corresponding value of  $M_p$  if subjected to a step input. For  $\zeta > 1/\sqrt{2}$ , the resonant peak  $M_r$  does not exist and the correlation breaks down.

Similarly, the expression for damped natural frequency for a second order system is given as

$$\omega_{1d} = \omega_{1n} [1 - (\zeta^2)]^{1/2}$$

Thus, there exists definite correlation between  $\omega_r$  of the frequency response and damped frequency of oscillation of the step response.

$$\omega_r / \omega_{1d} = \sqrt{[(1 - 2(\zeta^2)) / (1 - (\zeta^2))]}$$



Mr, Mp versus ζ

It is further observed that the bandwidth, a frequency domain concept, is indicative of the un-damped natural frequency of a system for a given  $\zeta$ , and therefore indicative of the speed of response ( $t_s = 4 / (\omega_n)$ ), a time-domain concept.

**Commonly used frequency response analysis Methods:**

Commonly used frequency response analysis Methods are:

- Bode plot

- Nyquist plot
- Nichols chart

Bode plot consists of two simultaneous graphs:

- Magnitude in dB  $[(20 \log |G(j\omega)|)(\text{Base } 10)]$  vs. frequency (in  $\log \omega$ )
- Phase (in degrees) vs. frequency (in  $\log \omega$ )

In the logarithmic representation, the curves are drawn on semilog paper, using the log scale for frequency and the linear scale for either magnitude (in Decibels) or phase angle (in degrees).

### Advantages of Bode Plot:

- Multiplication of Magnitudes can be converted into addition
- A simple method of sketching Bode Plot is based on asymptotic approximations. Such information on straight line asymptotes is sufficient if only rough information on frequency- response characteristics is needed.
- Should the exact curve be desired, corrections can be made easily to these basic asymptotic plots.
- Low frequency response contains sufficient information about the physical characteristics of most of the practical systems.
- Experimental determination of a transfer function is possible through Bode plot analysis.

### Bode Diagrams

In Bode diagrams, frequency ratios are expressed in terms of:

- **Octave:** it is a frequency band from  $\omega_1$  to  $2\omega_1$ .
- **Decade:** it is a frequency band from  $\omega_1$  to  $10\omega_1$ , where  $\omega_1$  is any frequency value.

The basic factors which occur frequently in an arbitrary transfer function are:

- Gain K
- Integral and derivatives:  $(j\omega)^{\pm 1}$
- First order factors:  $(1 + j\omega T)^{\pm 1}$ ,  $T = \frac{1}{\omega_c}$
- Quadratic Factors:  $(1 + 2\zeta(j\omega\omega_n) + (j\omega\omega_n)^2)^{\pm 1}$

### Bode Diagrams

- For Constant Gain K, log-magnitude curve is a horizontal straight line at the magnitude of  $(20 \log K)$  dB and phase angle is 0 deg.
- Varying the gain K, raises or lowers the log-magnitude curve of the transfer function by the

corresponding constant amount, but has no effect on the phase curve

- Logarithmic representation of the frequency-response curve of factor  $(j(\omega/a) + 1)$  can be approximated by two straight-line asymptotes
- Frequency at which the two asymptotes meet is called the **corner frequency** or **break frequency**.

## The Gain K:

Magnitude Response:

$$\text{Log Magnitude} = 20 \log K$$

As a number increases by a factor of 10, the corresponding value increases by a factor of 20. This may be seen from the following:

$$20 \log(K \times 10) = 20 \log K + 20$$

$$20 \log(K \times 10^n) = 20 \log K + 20n$$

Again, when expressed in decibels, the reciprocal of a number differs from its value only in sign, i.e., for the number K,

$$20 \log K = -20 \log \frac{1}{K}$$

## Integral and Derivative Factors $(j\omega)^{-1}$

Log Magnitude Plot:

Log Magnitude of  $\frac{1}{j\omega}$  is

$$20 \log \left| \frac{1}{j\omega} \right| = -20 \log \omega$$

Phase Plot:

The phase angle of  $\frac{1}{j\omega}$  is constant and equal to  $-90^\circ$ .

If the log magnitude  $-20 \log \omega$  is plotted on a logarithmic scale, it is a straight line. To draw this straight line, we need to locate one point (0 dB,  $\omega = 1$ ) on it. Since

$$-20 \log 10\omega \text{ dB} = (-20 \log \omega - 20) \text{ dB},$$

The slope of the line is -20 dB/decade or -6 dB/octave.

Similarly,

Log Magnitude of  $j\omega$  is

$$= 20 \log \omega$$

The phase angle of  $j\omega$  is constant and equal to  $90^\circ$ .

It can be seen that the differences in the frequency responses of  $\frac{1}{j\omega}$  and  $j\omega$  lie in the slopes of the log-magnitude curves and in the signs of the phase angles.

If the transfer function contains the factor  $\left(\frac{1}{j\omega}\right)^n$  or  $(j\omega)^n$ , the log magnitude becomes respectively,

$$20 \log \left| \frac{1}{(j\omega)^n} \right| = -n \times 20 \log j\omega = -20n \log \omega \text{ dB} \quad \text{or}$$

$$20 \log |(j\omega)^n| = n \times 20 \log j\omega = 20n \log \omega \text{ dB}$$

The slopes of the log-magnitude curve for the factors  $\frac{1}{(j\omega)^n}$  and  $(j\omega)^n$  are thus  $-20n$  dB/decade and  $20n$  dB/decade respectively. The phase angle of  $\frac{1}{(j\omega)^n}$  is equal to  $-90^\circ n$  over the entire frequency range, whereas that of  $(j\omega)^n$  is  $90^\circ n$  over the entire frequency range. The magnitude curve will pass through the point (0 dB,  $\omega = 1$ )

### First-Order Factors $(1 + j\omega T)^{-1}$ :

#### Log-Magnitude Curve:

The log magnitude of the first order factor  $\frac{1}{1 + j\omega T}$  is  $20 \log \left| \frac{1}{1 + j\omega T} \right| = -20 \log \sqrt{1 + \omega^2 T^2}$  dB

For low frequencies, such that  $\omega \ll \frac{1}{T}$ , the log magnitude may be approximated by

$$-20 \log \sqrt{1 + \omega^2 T^2} = -20 \log 1 = 0 \text{ dB.}$$

Thus, the log magnitude curve at low frequencies is the constant 0-dB line. For high frequencies, such

that  $\omega \gg \frac{1}{T}$ ,  $-20 \log \sqrt{1 + \omega^2 T^2} \approx -20 \log \omega T = -20 \log \omega - 20 \log T$  dB.

At  $\omega = \frac{1}{T}$ , the log magnitude equals 0 dB; at  $\omega = \frac{10}{T}$ , the log magnitude is -20 dB. Thus, the value of  $-20 \log \omega T$  decreases by 20 dB for every decade of  $\omega$ . For  $\omega \gg \frac{1}{T}$ , the log-magnitude curve is thus a straight line with a slope of -20 dB/decade (or -6 dB/octave).

Our analysis shows that the logarithmic representation of the frequency-response curve for the factor  $\frac{1}{(1 + j\omega T)}$  can be approximated by two straight-line asymptotes, one a straight-line at 0 dB for the

frequency range  $0 < \omega < \frac{1}{T}$  and the other a straight line with slope -20 dB/decade for the frequency range

$$\frac{1}{T} < \omega < \infty$$

The frequencies at which the two asymptotes meet is called the *Corner Frequency* or the *Break Frequency*.

For the factor  $\frac{1}{1 + j\omega T}$ ,  $\omega = \frac{1}{T}$  is the corner frequency. The corner frequency thus divides the frequency-response curve into two regions: The low frequency region and the high frequency region.

## Phase Plot:

The exact phase angle  $\phi$  of the factor  $1/(1 + j\omega T)$  is  $\phi = \tan^{-1} \omega T$

At zero frequency, the phase angle is  $0^\circ$ . At the corner frequency, the phase angle is

$$\phi = -\tan^{-1}(-1) (T/T) = -\tan^{-1}(-1) 1 = -45^\circ$$

At infinite frequency, the phase angle becomes  $-90^\circ$ . Since the phase angle is given by an inverse tangent function, it is skew-symmetric about the inflection point at  $\phi = -45^\circ$ .

## Error in the Magnitude curve:

The error in the Magnitude curve caused by the use of asymptotes can be calculated.

Error at a particular frequency = Actual value – Approximate value of the log-magnitude curve at that frequency

The maximum error occurs at the corner frequency

$$\text{Actual value} = -20 \log \sqrt{1 + 1} = -10 \log 2 = -3.03 \text{ dB}$$

$$\text{Approximate value} = -20 \log 1 = 0 \text{ dB.}$$

Thus, error at corner frequency = -3 dB.

The error at one octave below the corner frequency, i.e., at  $\omega = \frac{1}{2T}$  is

$$-20 \log \sqrt{1 + \frac{1}{4}} + 20 \log 1 = -20 \log \left[ \frac{\sqrt{5}}{2} \right] = -0.97 \text{ dB}$$

The error at one octave above the corner frequency, i.e., at  $\omega = \frac{2}{T}$  is

$$-20 \log \sqrt{2^2 + 1} + 20 \log 2 = -20 \log \left[ \frac{\sqrt{5}}{2} \right] = -0.97 \text{ dB}$$

Thus, the error at one octave above or below the corner frequency is approximately -1 dB.

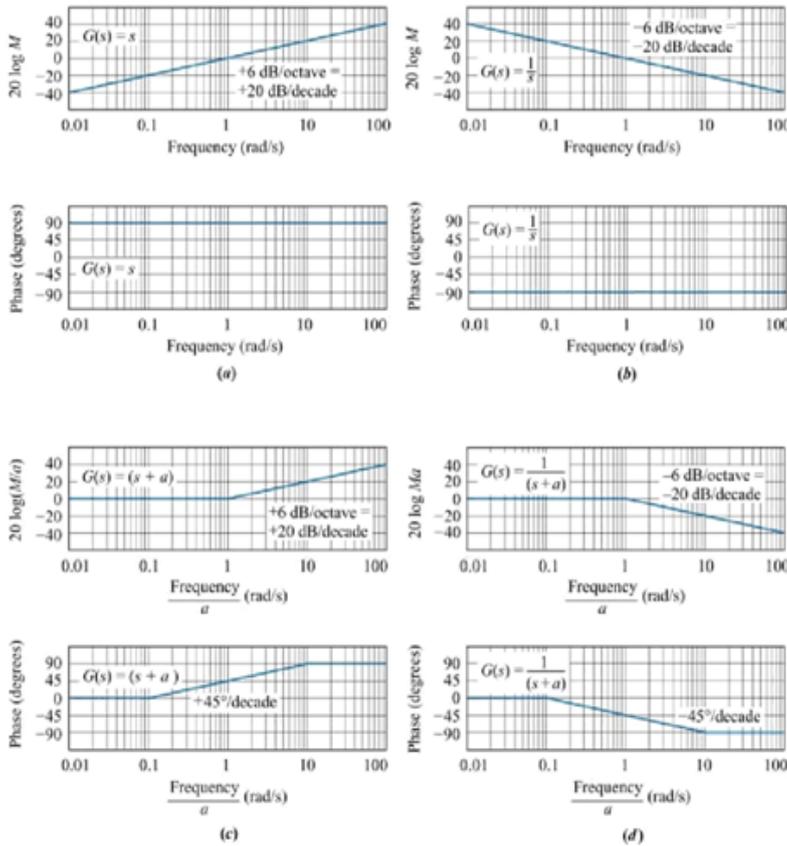
The transfer function  $\frac{1}{1 + j\omega T}$  has the characteristics of a low-pass filter. For frequencies above  $\omega = \frac{1}{T}$ , the log-magnitude falls rapidly towards  $-\infty$ .

# Bode diagrams of some standard first order terms

Ref: N. S. Nise,  
Control Systems  
Engineering, 4<sup>th</sup> Ed.  
Wiley, 2004.

Bode plots for:

- a.  $G(s) = s$
- b.  $G(s) = 1/s$
- c.  $G(s) = (s + a)$
- d.  $G(s) = 1/(s + a)$



**Quadratic Factors**  $\left[ 1 + 2\zeta \left( j \frac{\omega}{\omega_n} \right) + \left( j \frac{\omega}{\omega_n} \right)^2 \right]^{-1}$  :

When there are complex conjugate zeroes, the prototype 2<sup>nd</sup> order systems will have the transfer function

$$G(s) = s^2 + 2\zeta\omega_n s + \omega_n^2, \quad 0 < \zeta < 1$$

When there are complex conjugate poles, the prototype 2<sup>nd</sup> order systems will have the transfer function

$$G(s) = \frac{1}{(s^2 + 2\zeta\omega_n s + \omega_n^2)}, \quad 0 < \zeta < 1$$

For the complex conjugate poles,

$$G(j\omega) = \frac{1}{\left[ 1 + 2\zeta \left( j \frac{\omega}{\omega_n} \right) + \left( j \frac{\omega}{\omega_n} \right)^2 \right]}$$

Log Magnitude Curve:

$$\text{Log magnitude} = 20 \log \left| \frac{1}{\left[1 + 2\zeta \left(j\frac{\omega}{\omega_n}\right) + \left(j\frac{\omega}{\omega_n}\right)^2\right]} \right| = -20 \log \sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\zeta \left(j\frac{\omega}{\omega_n}\right)\right)^2}$$

For low frequencies, i.e.,  $\ll \omega_n$ ,

Log magnitude becomes  $-20 \log 1 = 0$  dB

The low frequencies asymptote is thus a horizontal line at 0 dB.

For high frequencies i.e.,  $\gg \omega_n$ ,

$$\text{Log magnitude becomes } -20 \log \left(\frac{\omega}{\omega_n}\right)^2 = -40 \log \frac{\omega}{\omega_n} \text{ dB} = -40 - 40 \log \frac{\omega}{\omega_n} \text{ dB.}$$

The high frequency asymptote is thus a straight line having the slope -40 dB/ decade.

The high frequency asymptote intersects the low-frequency one at  $\omega = \omega_n$ , the corner frequency.

The two asymptotes derived are independent of  $\zeta$ . The resonant peak occurs near the frequency  $\omega = \omega_n$ .

The damping ration  $\zeta$  determines the magnitude of this resonant peak. The magnitude of errors caused by the straight line asymptotes depend on the value of  $\zeta$ . It is large for small values of  $\zeta$ .

**Phase Plot:**

The phase angle of the quadratic factor  $\frac{1}{\left[1 + 2\zeta \left(j\frac{\omega}{\omega_n}\right) + \left(j\frac{\omega}{\omega_n}\right)^2\right]}$  is

$$\phi = \angle \frac{1}{\left[1 + 2\zeta \left(j\frac{\omega}{\omega_n}\right) + \left(j\frac{\omega}{\omega_n}\right)^2\right]} = -\tan^{-1} \left[ \frac{2\zeta \left(\frac{\omega}{\omega_n}\right)}{1 - \left(\frac{\omega}{\omega_n}\right)^2} \right]$$

The phase angle is a function of both  $\zeta$  and  $\omega$ .

$$\omega = 0, \quad \phi = 0^\circ$$

$$\omega = \omega_n \quad \phi = -\tan^{-1}(-1) (2\zeta/0) = \tan^{-1}(-1) \infty = -90^\circ$$

$$\omega = \infty, \quad \phi = -180^\circ$$

The phase angle curve is skew-symmetric about the inflection point where  $\phi = -90^\circ$ .

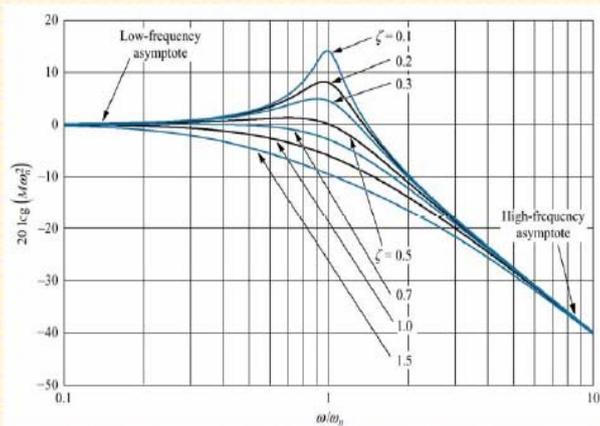
The frequency response for the factor

$$G(j\omega) = \left[1 + 2\zeta \left(j\frac{\omega}{\omega_n}\right) + \left(j\frac{\omega}{\omega_n}\right)^2\right]$$

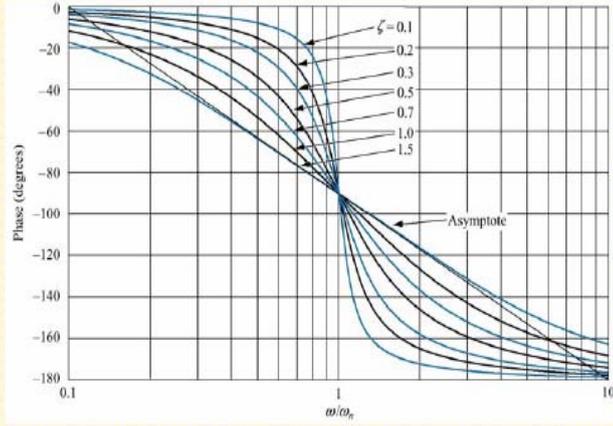
Can be obtained by merely reversing the sign of the log magnitude and that of the phase angle for the factor

$$G(j\omega) = \frac{1}{\left[1 + 2\zeta \left(j\frac{\omega}{\omega_n}\right) + \left(j\frac{\omega}{\omega_n}\right)^2\right]}$$

## Scaled Response for $1/(s^2 + 2\zeta\omega_n s + \omega_n^2)$



Magnitude Plot



Phase Plot

### Relationship between System Type and Log-Magnitude Curve:

For a unity feedback system the static position, velocity and acceleration error constants describe the low-frequency behavior of type 0, type 1, and type 2 systems respectively. For a given system, only one of the static error constants is finite and significant. (The larger the value of the finite static error constant, the higher the loop gain is as  $\omega$  approaches zero.)

The type of the system determines the slope of the log-magnitude curve at low frequencies. Thus, information concerning the existence and magnitude of the steady-state error of a control system to a given input can be determined from the observation of the low-frequency region of the log-magnitude curve.

### Determination of Static Error constants:

Assume that the open loop transfer function of a unity feedback system is given by

$$G(s) = \frac{K(T_{a_1}s + 1)(T_{a_2}s + 1) \dots (T_{a_m}s + 1)}{s^N(T_{b_1}s + 1)(T_{b_2}s + 1) \dots (T_{b_p}s + 1)}$$

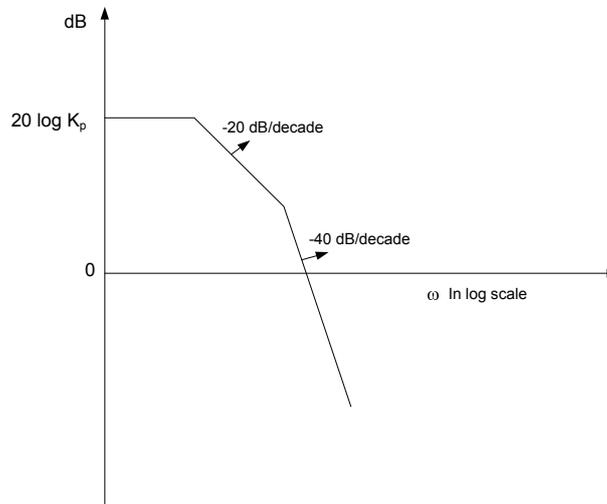
Or

$$G(j\omega) = \frac{K(T_{a_1}j\omega + 1)(T_{a_2}j\omega + 1) \dots (T_{a_m}j\omega + 1)}{(j\omega)^N(T_{b_1}j\omega + 1)(T_{b_2}j\omega + 1) \dots (T_{b_p}j\omega + 1)}$$

## Static Position Error constant:

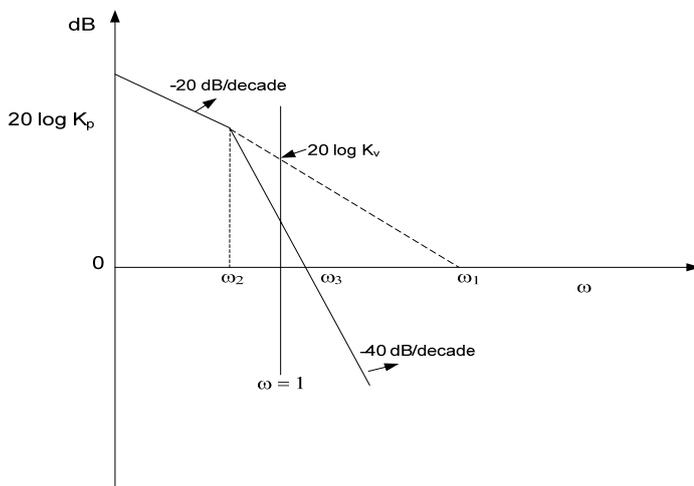
The figure shown below shows an example of the log-magnitude plot of a type 0 system. In such a system, the magnitude of  $G(j\omega)$  equals  $K_p$  at low frequencies, or

$$\lim_{\omega \rightarrow 0} G(j\omega) = K = K_p$$



## Static Velocity Error constant:

The figure given below shows an example of the log-magnitude of a type 1 unity feedback system. The intersection of the initial -20 dB/decade segment (or its extension) with the line  $\omega = 1$  has the magnitude  $20 \log K_v$ . This may be seen as follows.



In a type-1 system,

$$G(j\omega) = \frac{K_v}{j\omega}, \text{ for } \omega \ll 1.$$

Thus,

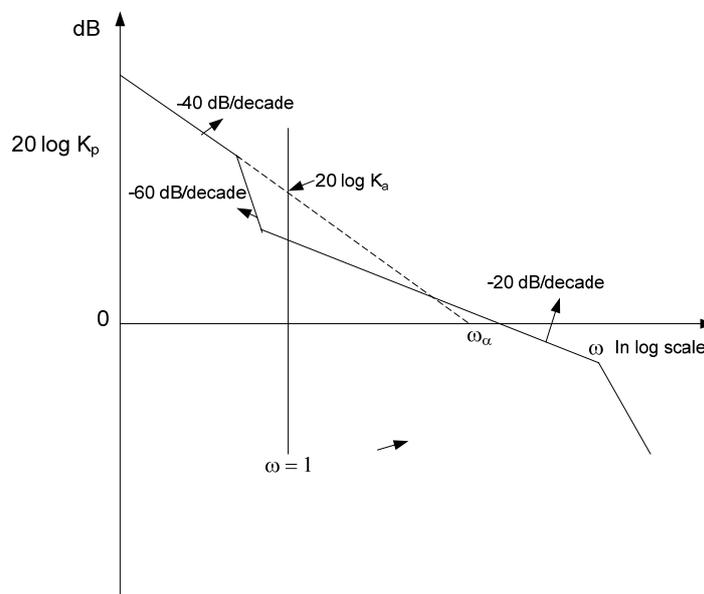
$$20 \log \left( \frac{K_v}{j\omega} \right)_{\omega=1} = 20 \log K_v$$

The intersection of the initial -20 dB/decade segment (or its extension) with the 0-dB line has a frequency numerically equal to  $K_v$ , i.e., if the frequency at this intersection is  $\omega_1$ , then

$$\left| \frac{K_v}{j\omega} \right| = 1 \quad \text{or,} \quad K_v = \omega_1$$

### Static Acceleration Error constant:

The figure given below shows an example of the log-magnitude of a type 2 unity feedback system. The intersection of the initial -40 dB/decade segment (or its extension) with the line  $\omega = 1$  has the magnitude  $20 \log K_a$ . This may be seen as follows.



Since at low frequencies,

$$G(j\omega) = \frac{K_a}{(j\omega)^2}, \text{ for } \omega \ll 1.$$

It follows that

$$20 \log \left( \frac{K_a}{(j\omega)^2} \right)_{\omega=1} = 20 \log K_a$$

The frequency  $\omega_\alpha$  at the intersection of the initial -40 dB/decade segment (or its extension) with the 0-dB line gives the square root of  $K_a$  numerically. This can be seen from the following.

$$20 \log \left| \frac{K_a}{(j\omega_\alpha)^2} \right| = 20 \log 1 = 0$$

Which yields 
$$\omega_\alpha = \sqrt{K_a}$$

## Phase Margin (PM):

Phase margin is that amount of additional phase lag at the gain crossover frequency required to bring the system to the verge of instability.

## Gain Crossover Frequency:

Gain crossover frequency is that frequency at which,  $|G(j\omega)|$ , the magnitude of the open loop transfer function is unity.

The Phase margin PM is  $180^\circ$  plus the phase angle  $\phi$  of the open loop transfer function at the gain crossover frequency.

$$PM = (180^\circ + \phi)_{at\ GCF}$$

## Gain Margin (PM):

Gain Margin is the reciprocal of the magnitude  $|G(j\omega)|$  at the Phase crossover frequency.

## Phase Crossover Frequency:

Phase crossover frequency is that frequency at which,  $\angle G(j\omega)$ , the phase angle of the open loop transfer function equals  $-180^\circ$ .

Thus, Gain Margin,

$$GM = \left( \frac{1}{|G(j\omega)|} \right)_{at\ PCF}$$

In terms of decibels,

$$GM\ dB = 20 \log |G(j\omega)|_{at\ PCF}$$

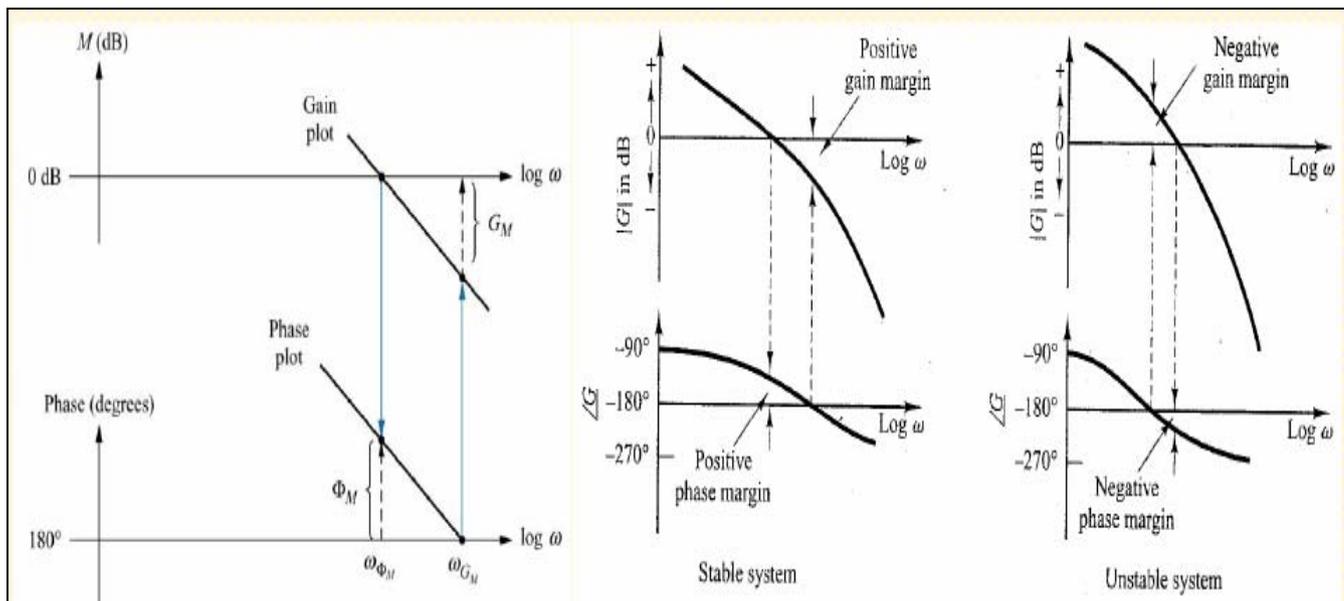
## A Few Comments on Phase and Gain Margins:

- For a stable non-minimum phase system, the gain margin indicates how much the gain can be increased before the system becomes unstable. For an unstable system, the gain margin indicates how much the gain can be decreased before the system becomes stable.
- The Gain Margin of a first and Second order system is infinite since the polar plot of such systems does not cross the real axis. Thus, theoretically, the 1<sup>st</sup> and 2<sup>nd</sup> order systems cannot be unstable.
- It is important to point out that conditionally stable systems will have two or more phase crossover frequencies and some higher order systems with complicated numerator dynamics may also have two or more gain crossover frequencies. For stable systems having two or more gain crossover frequencies, the Phase Margin is measured at the highest Gain Cross-over Frequency.
- Either the Gain Margin alone or the Phase Margin alone does not give a sufficient indication of the relative stability. Both should be given for determination of stability.

# Control Systems

- For satisfactory performance, PM should be between  $30^\circ$  and  $60^\circ$  and the GM should be greater than 6 dB.
- The requirement that the PM be between  $30^\circ$  and  $60^\circ$  means that in Bode diagram, the slope of the log-magnitude curve at the gain crossover frequency should be more gradual than  $-40$  dB/decade. In most practical cases, a slope of  $-20$  dB/decade is desirable. If the slope at the gain crossover frequency is  $-60$  dB/decade or steeper, the system is most likely unstable.

## PHASE AND GAIN MARGIN THROUGH BODE PLOTS:

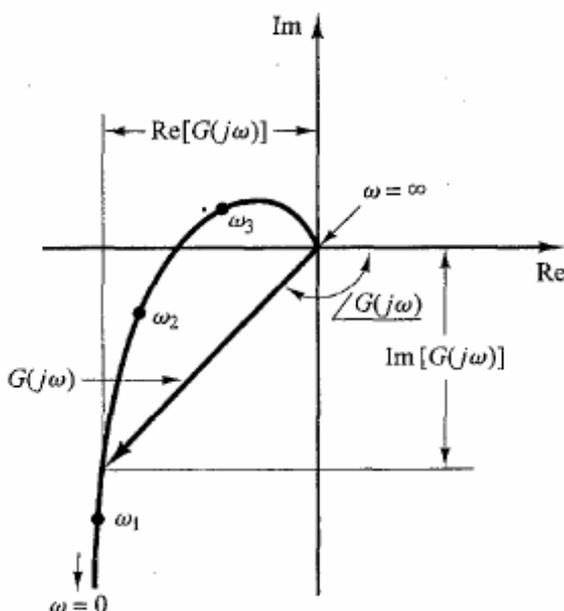


A sinusoidal transfer function  $G(j\omega)$  is a complex function and is given by

$$G(j\omega) = \text{Re}[G(j\omega)] + j \text{Im}[G(j\omega)]$$

Or, 
$$G(j\omega) = |G(j\omega)| \angle G(j\omega) = M \angle \phi$$

It is seen that  $G(j\omega)$  can be represented as a phasor of magnitude  $M$  and phase angle  $\phi$  (Measured positively in counter-clockwise direction). As the input frequency  $\omega$  is varied from 0 to  $\infty$ , the magnitude  $M$  and the phase angle  $\phi$  change and hence the tip of the phasor  $G(j\omega)$  traces a locus in the complex plane. The locus thus obtained is known as 'Polar Plot' as shown below.



### Procedure for Sketching of the Polar Plot:

To sketch the Polar Plot of a given Open Loop Transfer Function over the entire frequency range,

- Express the given expression for the OLTF in  $(1+sT)$  form.
- Substitute  $s = j\omega$  in the expression for  $G(s)H(s)$  and get  $G(j\omega)H(j\omega)$ .
- Find out the expressions for  $|G(j\omega)H(j\omega)|$  and  $\angle G(j\omega)H(j\omega)$ .
- Tabulate various values of magnitude and phase angle for different values of  $\omega$  starting from 0 to  $\infty$ .
- There are usually four key points to be known.
  - (a) The starting of the plot where  $\omega = 0$ .
  - (b) The end of the plot where  $\omega = \infty$ .
  - (c) The point where the Polar plot crosses the real axis, i.e.,  $\text{Im}(G(j\omega)) = 0$
  - (d) The point where the Polar plot crosses the imaginary axis, i.e.,  $\text{Re}(G(j\omega)) = 0$ .

# Control Systems

- Fix all points in a polar graph sheet and join the points. (Polar graph sheet has concentric circles and radial lines. The concentric circles represent the magnitude and the radial lines represent the phase angles. In polar sheet, +ve phase angle is measured in ACW from  $0^\circ$  and -ve phase angle is measured in CW from  $0^\circ$ )

Examples:

Polar Plot of  $G(s) = \frac{1}{1+sT}$ :

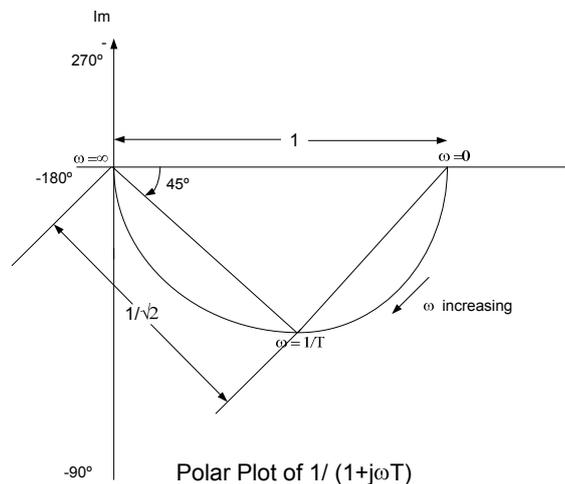
Consider a 1<sup>st</sup> order system with transfer function  $G(s) = \frac{1}{1+sT}$

The sinusoidal transfer function is

$$G(j\omega) = \frac{1}{1+j\omega T}$$

$$(1 + \omega^2 T^2)^{-1/2} \angle -\tan^{-1} \omega T = M \angle \phi$$

When  $\omega = 0, M = 1$  and  $\phi = 0$ . Therefore, the phasor at  $\omega = 0$  has unit length and lies along the positive real axis. As  $\omega$  increases, M decreases and phase angle increases negatively. When  $\omega = \frac{1}{T}$ ,  $M = \frac{1}{\sqrt{2}}$  and  $\phi = -45^\circ$ . As  $\omega \rightarrow \infty$ , M becomes zero and  $\phi$  is  $-90^\circ$ . This is represented by a phasor of zero length directed along the  $-90^\circ$  axis in the complex plane. In fact, the locus of  $G(j\omega)$  can be shown to be a semicircle.



Polar Plot of  $G(s) = \frac{1}{s(1+sT)}$ :

Consider now the transfer function

$$G(j\omega) = \frac{1}{j\omega(1+j\omega T)}$$

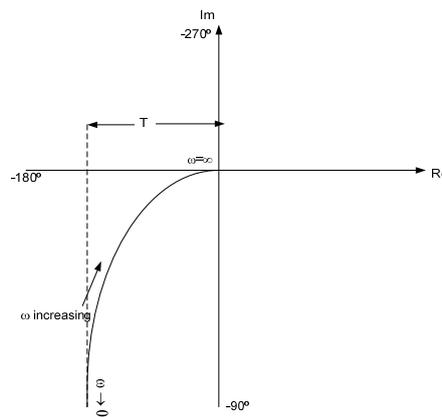
This transfer function may be rearranged as

$$G(j\omega) = \frac{-T}{1 + \omega^2 T^2} - j \frac{1}{\omega(1 + \omega^2 T^2)}$$

$$\lim_{\omega \rightarrow 0} G(j\omega) = -T - j\infty = \infty \angle -90^\circ$$

$$\lim_{\omega \rightarrow \infty} G(j\omega) = -0 - j0 = 0 \angle -180^\circ$$

The general shape of this transfer function is shown below. The plot is asymptotic to the vertical line passing through the point  $(-T, 0)$ .



Polar Plot of  $1/j\omega(1+j\omega T)$

Polar Plot of  $G(s) = \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2}$ :

The low and high frequency regions of the polar plot of the following sinusoidal transfer function

$$G(j\omega) = \frac{1}{\left[1 + 2\zeta \left(j\frac{\omega}{\omega_n}\right) + \left(j\frac{\omega}{\omega_n}\right)^2\right]}$$

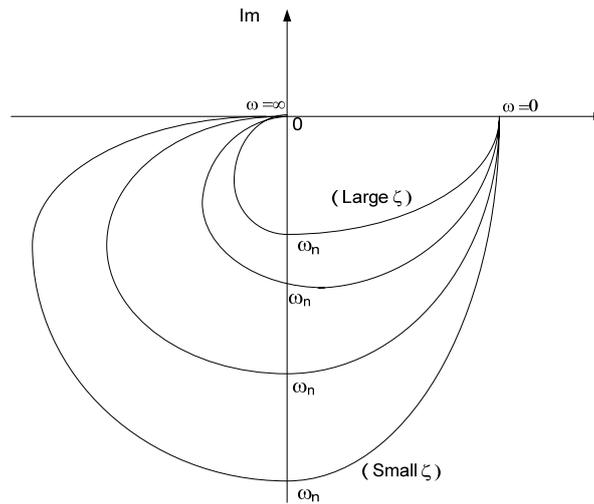
are given respectively by

$$\lim_{\omega \rightarrow 0} G(j\omega) = 1 \angle 0^\circ$$

and

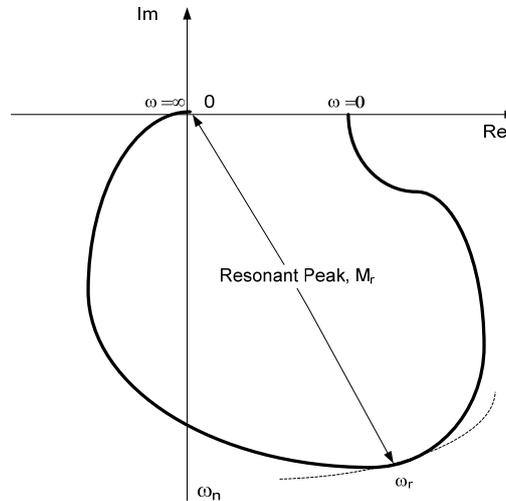
$$\lim_{\omega \rightarrow \infty} G(j\omega) = 0 \angle -180^\circ$$

The Polar plot of this sinusoidal transfer function starts at  $1 \angle 0^\circ$  and ends at  $0 \angle -180^\circ$  as  $\omega$  increases from zero to infinity. Thus, the high frequency portion of  $G(j\omega)$  is a tangent to the negative real axis.



The exact shape of a polar plot depends on the value of the damping ratio  $\zeta$ . Thus the general shape is same for both overdamped and underdamped case.

For the under-damped case, at  $\omega = \omega_n$ , we have  $G(j\omega_n) = \frac{1}{2\zeta}$ , and the phase angle at  $\omega = \omega_n$  is  $-90^\circ$ . Therefore, it can be seen that the frequency at which the  $G(j\omega)$  locus intersects the imaginary axis is the undamped natural frequency  $\omega_n$ . The peak value of  $G(j\omega)$  is obtained as the ratio of magnitude of the vector at the resonant frequency  $\omega_r$  to the magnitude of the vector at  $\omega = 0$ .



For the over-damped case, as  $\zeta$  increases well beyond unity, the  $G(j\omega)$  locus approaches a semi-circle. This may be seen from the fact that, for a heavily damped system, the characteristic roots are real, and one is much smaller than the other. Since, for sufficiently large  $\zeta$ , the effect of the larger root (larger in absolute value) on the response becomes very small, the system behaves like a 1<sup>st</sup> order one.

**Example:**

Obtain the Polar Plot of the following transfer function:

$$G(j\omega) = \frac{e^{-j\omega L}}{1 + j\omega T}$$

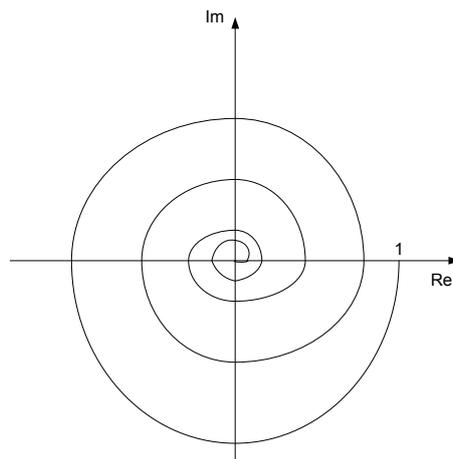
Since  $G(j\omega)$  can be written as  $(e^{-j\omega L})\left(\frac{1}{1 + j\omega T}\right)$

The magnitude and phase angle are respectively

$$|G(j\omega)| = |e^{-j\omega L}| \left| \frac{1}{1 + j\omega T} \right| = \frac{1}{\sqrt{1 + \omega^2 T^2}}$$

And

$$\angle G(j\omega) = \angle e^{-j\omega L} + \angle \frac{1}{1 + j\omega T} = -\omega L - \tan^{-1} \omega T$$



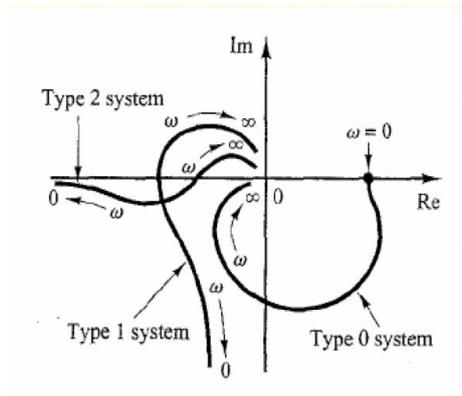
Since the magnitude decreases from unity monotonically, and the phase angle also decreases monotonically, and indefinitely, the polar plot of the given transfer function is a spiral, as shown in the above figure.

**General Nature of Nyquist Plots:**

The polar plots of a transfer function of the form

$$G(j\omega) = \frac{K(1 + T\omega_1 j\omega)(1 + T\omega_2 j\omega) \dots (1 + T\omega_m j\omega)}{(j\omega)^N (1 + T\omega_{1'} j\omega)(1 + T\omega_{2'} j\omega) \dots (1 + T\omega_{p'} j\omega)}$$

Where,  $n > m$  or the degree of the denominator polynomial is greater than that of the numerator will have the following shapes.



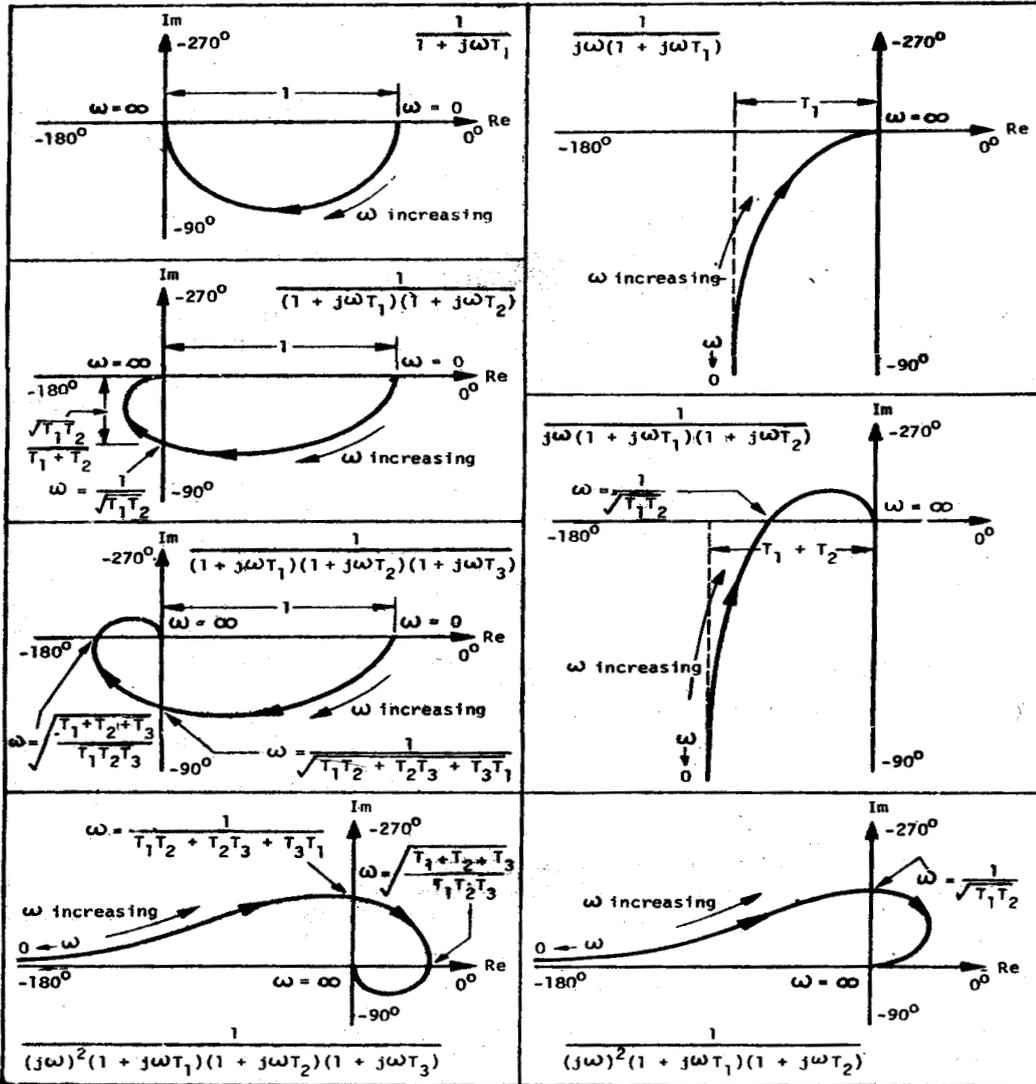
## General shapes of the polar plots of some important functions

The general shapes of the polar plots of some important functions are shown below. From the figures, following observations are made.

Addition of a non-zero pole to a transfer function results in further rotation of the polar plot through an angle of  $-90^\circ$  as  $\omega \rightarrow \infty$ .

- Addition of a pole at the origin to the transfer function rotates the polar plot at zero and infinite frequencies by a further angle of  $-90^\circ$ .
- The effect of addition of a zero to the transfer function is to rotate the high frequency portion of the polar plot by  $90^\circ$  in the counter-clockwise direction.
- If degree of the denominator polynomial is greater than that of the numerator, the  $G(j\omega)$  loci will converge to the origin clockwise.
- Any complicated shape of the polar plot curves are caused by the numerator dynamics, which is by the time constants in the numerator of the transfer function.

## General shapes of the polar plots of some other important functions



### RELATIVE STABILITY ANALYSIS:

In designing a control system, we require that the system be stable. We also require that the system has adequate relative stability. The closeness of approach of the  $G(j\omega)$  locus to the  $-1+j0$  point is an indication of the relative stability of a stable system. In general, we may expect that the closer the  $G(j\omega)$  locus is to the  $1+j0$  point, the larger is the maximum overshoot in the transient response and the larger it takes to damp out. When the  $G(j\omega)$  locus passes through the  $1+j0$  point, the system is on the verge of instability and exhibits sustained oscillations. The measures of relative stability in the frequency domain are the Gain Margin and the Phase Margin.

## 10. Stability in frequency domain:

A stability test for time invariant linear systems can also be derived in the frequency domain. It is known as Nyquist stability criterion.

It is based on the complex analysis result known as Cauchy's principle of argument. Note that the system transfer function is a complex function. By applying Cauchy's principle of argument to the open-loop system transfer function, we will get information about stability of the closed-loop system transfer function and arrive at the Nyquist stability criterion (Nyquist, 1932).

The importance of Nyquist stability lies in the fact that it can also be used to determine the relative degree of system stability by producing the so-called phase and gain stability margins. These stability margins are needed for frequency domain controller design techniques.

We present Only the Essence of the Nyquist stability Criterion and Define the Phase and Gain stability margins. The Nyquist Method is used for studying the stability of linear Systems with Pure time delay.

For a SISO feedback System the closed-loop transfer function is given by:

$$M(s) = \frac{G(s)}{1 + H(s)G(s)}$$

where  $G(s)$  represents the system and  $H(s)$  is the feedback element.

Since the system poles are determined as those values at which its transfer function becomes infinity, it follows that the closed-loop system poles are obtained by solving the following equation

$$1 + H(s)G(s) = 0 = \Delta(s)$$

which, in fact, represents the System characteristic equation.

In the following we consider the complex function

$$D(s) = 1 + H(s)G(s)$$

Whose zeros are the closed-loop poles of the transfer function. In addition, it is easy to see that the poles of  $D(s)$  are the zeros of  $M(s)$ . At the same time the poles of  $D(s)$  are the open-loop control system poles since they are contributed by the poles of  $H(s)G(s)$ , which can be considered as the open-loop control system transfer function- obtained when the feedback loop is open at some point. The Nyquist stability test is obtained by applying the Cauchy principle of argument to the complex function  $D(s)$ . First, we state Cauchy's principle of argument.

### Cauchy's principle of argument

Let  $F(s)$  be an analytic function in a closed region of the complex plane  $s$  given in Figure below except at a finite number of points (namely, the poles of  $F(s)$ ). It is also assumed that  $F(s)$  is analytic at every point on the contour. Then, as  $s$  travels around the contour in the  $s$ -plane in the clockwise direction, the function  $F(s)$  encircles the origin in the  $(\text{Re}\{F(s)\}, \text{Im}\{F(s)\})$ -plane in the same direction  $N$  times (see Figure 4.6), with  $N$  given by

$$N = P - Z$$

# Control Systems

Where  $Z$  and  $P$  stand for the number of zeros and poles (including their multiplicities) of the function  $F(s)$  inside the contour.

The above result can be also written as

$$\arg\{F(s)\} = (Z - P)2\pi = 2\pi N$$

Which justifies the terminology used, "the principle of argument".

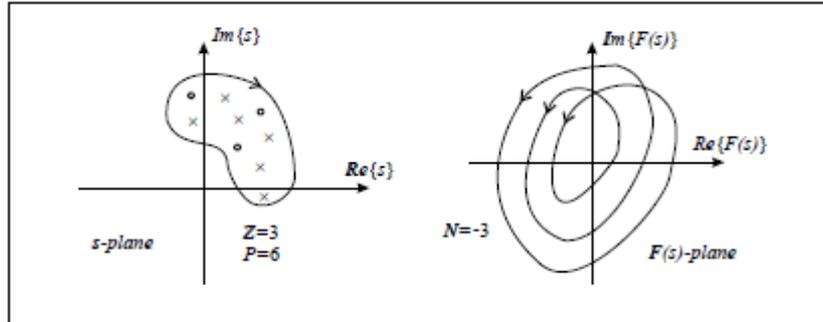


Figure 1. Cauchy's principle of argument

## Nyquist Plot

The Nyquist plot is a polar plot of the function  $D(s) = 1 + G(s)H(s)$

When  $s$  travels around the contour given in Figure below.

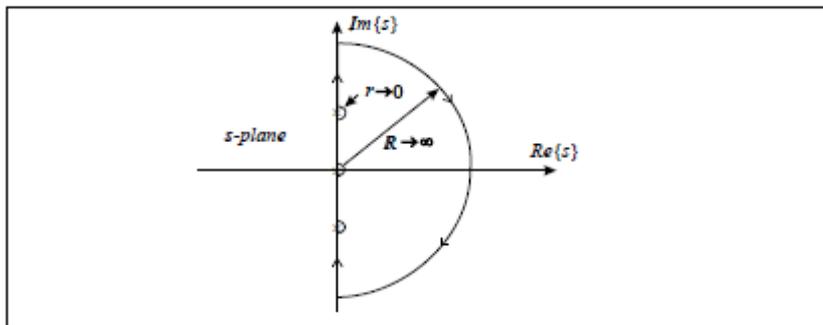


Figure 2. Contour in S-plane

The contour in this figure covers the whole unstable half plane of the complex plane  $s$ ,  $R \rightarrow \infty$ . Since the function  $D(s)$ , according to Cauchy's principle of argument, must be analytic at every point on the contour, the poles of  $D(s)$  on the imaginary axis must be encircled by infinitesimally small semicircles.

## Nyquist Stability Criterion

It states that the number of unstable closed-loop poles is equal to the number of unstable open-loop poles plus the number of encirclements of the origin of the Nyquist plot of the complex function.

This can be easily justified by applying Cauchy's principle of argument to the function with the  $s$ -plane contour given in Figure 2. Note that  $Z$  and  $P$  represent the numbers of zeros and poles, respectively, of in the unstable part of the complex plane. At the same time, the zeros of  $D(s)$  are the closed-loop system poles, and the poles of  $D(s)$  are the open-loop system poles (closed-loop zeros).

The above criterion can be slightly simplified if instead of plotting the function, we plot only the function  $G(s)H(s)$  and count encirclement of the Nyquist plot of around the point  $-1$ , so that the modified Nyquist criterion has the following form.

## Stability via the Nyquist Diagram

We now use the Nyquist diagram to determine a system's stability, using the simple equation. The values of  $P$ , the number of open-loop poles of  $G(s)H(s)$  enclosed by the contour, and  $N$ , the number of encirclements the Nyquist diagram makes about  $-1$ , are used to determine  $Z$ , the number of right-half-plane poles of the closed-loop system.

If the closed-loop system has a variable gain in the loop, one question we would like to ask is, "For what range of gain is the system stable?" The general approach is to set the loop gain equal to unity and draw the Nyquist diagram. Since gain is simply a multiplying factor, the effect of the gain is to multiply the resultant by a constant anywhere along the Nyquist diagram.

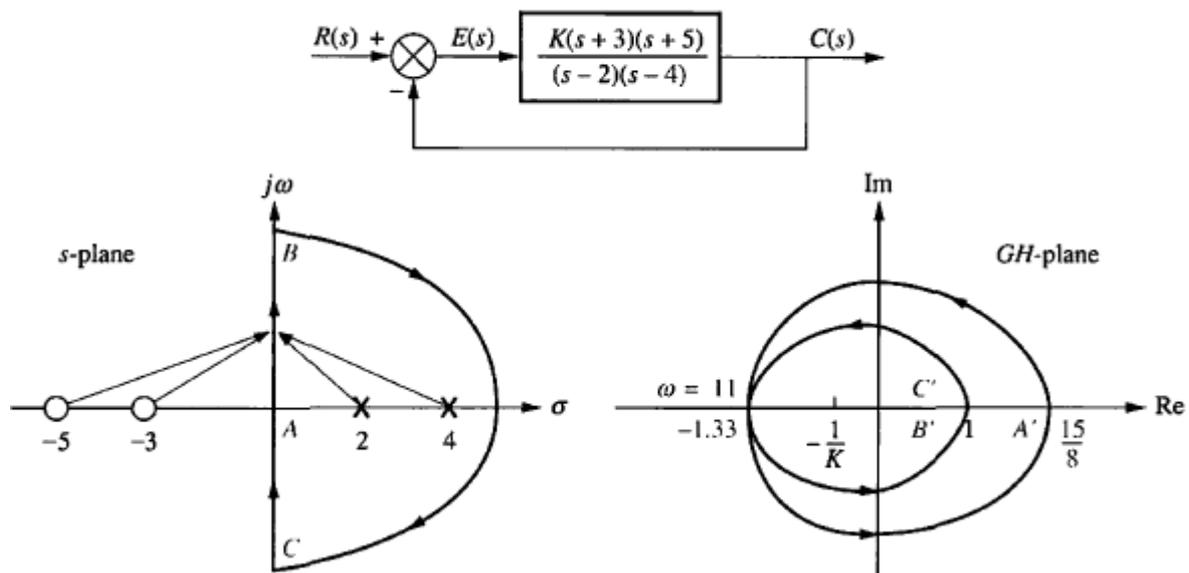


Figure 3. Nyquist stability contour and diagram

As the gain is varied, we can visualize the Nyquist diagram is expanding (increased gain) or shrinking (decreased gain) like a balloon. This motion could move the Nyquist diagram past the  $-1$  point, changing the stability picture. For this system, since  $P = 2$ , the critical point must be encircled by the Nyquist diagram to yield  $N = 2$  and a stable system. A reduction in gain would place the critical point outside the Nyquist diagram where  $N = 0$ , yielding  $Z = 2$ , an unstable system.

If the Nyquist diagram intersects the real axis at  $-1$ , then  $G(j\omega)H(j\omega) = -1$ . From root locus concepts, when  $G(s)H(s) = -1$ , the variable  $s$  is a closed-loop pole of the system. Thus, the frequency at which the Nyquist diagram intersects  $-1$  is the same frequency at which the root locus crosses the  $j\omega$ -axis. Hence, the system is marginally stable if the Nyquist diagram intersects the real axis at  $-1$ .

In summary, then, if the open-loop system contains a variable gain,  $K$ , set  $K = 1$  and sketch the Nyquist diagram. Consider the critical point to be at  $-1/K$  rather than at  $-1$ . Adjust the value of  $K$  to yield stability, based upon the Nyquist criterion.

**PROBLEM:** For the unity feedback system, where  $G(s) = K/[s(s + 3)(s + 5)]$ , find the range of gain,  $K$ , for stability, instability, and the value of gain for marginal stability. For marginal stability also find the frequency of oscillation. Use the Nyquist criterion.

**SOLUTION:** First set  $K = 1$  and sketch the Nyquist diagram for the system –

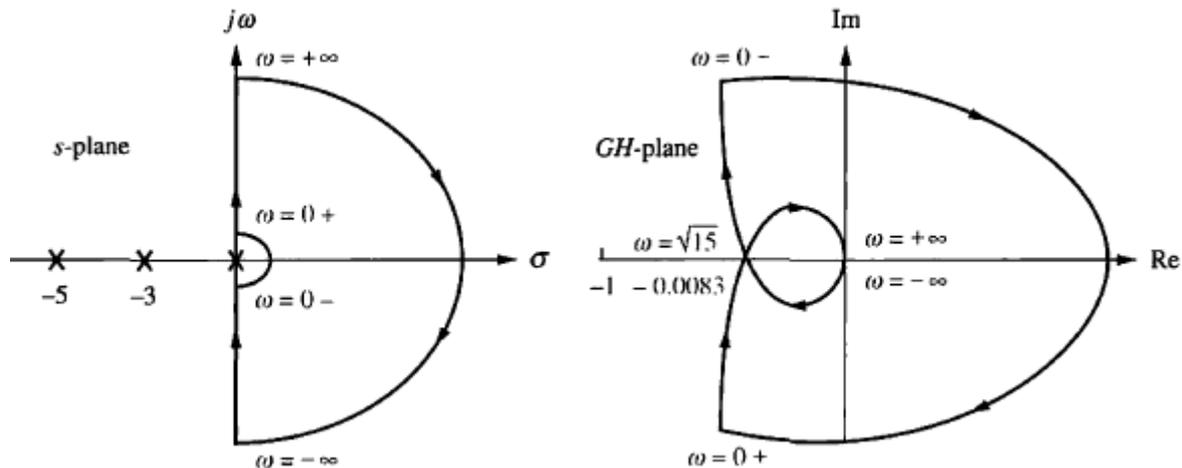


Figure 4. Nyquist contour and stability diagram

For all points on the imaginary axis,

$$G(j\omega)H(j\omega) = \frac{K}{s(s+3)(s+5)} \Big|_{s=j\omega}^{K=1} = \frac{-8\omega^2 - j(15\omega - \omega^3)}{64\omega^4 + \omega^2(15 - \omega^2)^2} \quad (1)$$

At  $\omega = 0$ ,  $G(j\omega)H(j\omega) = -0.0356 - j\infty$

Next find the point where the Nyquist diagram intersects the negative real axis. Setting the imaginary part of Eq. (1) equal to zero, we find  $\omega = \sqrt{15}$ .

Substituting this value of  $\omega$  back into Eq. (1) yields the real part of -0.0083. Finally, at  $\omega = \infty$ ,

$$G(j\omega)H(j\omega) = G(s)H(s) \Big|_{s \rightarrow j\infty} = \frac{1}{(j\infty)^3} = 0 \angle (-270^\circ).$$

From the contour of Figure,  $P = 0$ ; for stability  $N$  must then be equal to zero. From Figure, the system is stable if the critical point lies outside the contour ( $N = 0$ ), so that  $Z = P - N = 0$ . Thus,  $K$  can be increased by  $1/0.0083 = 120.5$  before the Nyquist diagram encircles  $-1$ .

Hence, for stability,  $K < 120.5$ . For marginal stability  $K = 120.5$ . At this gain the Nyquist diagram intersects  $-1$ , and the frequency of oscillation is  $\sqrt{15}$  rad/s

## Stability via Mapping Only the Positive $j\omega$ - Axis

Once the stability of a system is determined by the Nyquist criterion, continued evaluation of the system can be simplified by using just the mapping of the positive  $j\omega$  -axis.

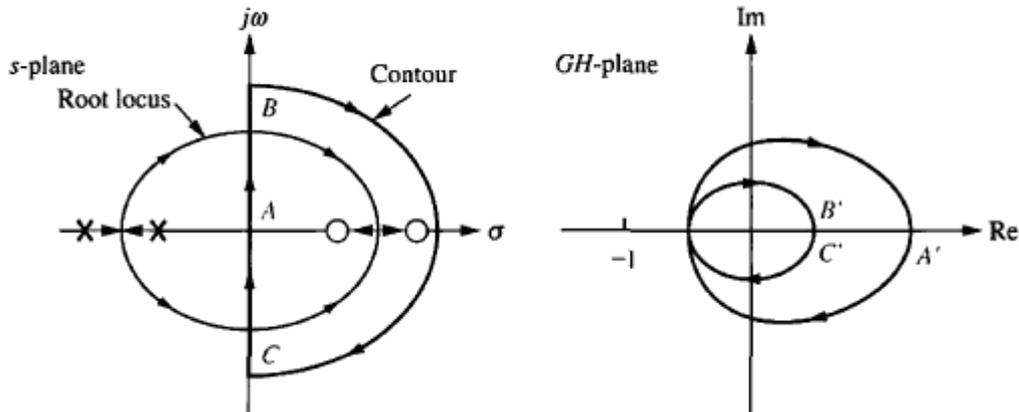


Figure 5. Contour and root locus of system that is stable for small gain and unstable for large gain

Consider the system shown in above Figure, which is stable at low values of gain and unstable at high values of gain. Since the contour does not encircle open-loop poles, the Nyquist criterion tells us that we must have no encirclements of  $-1$  for the system to be stable. We can see from the Nyquist diagram that the encirclements of the critical point can be determined from the mapping of the positive  $j\omega$  -axis alone. If the gain is small, the mapping will pass to the right of  $-1$ , and the system will be stable. If the gain is high, the mapping will pass to the left of  $-1$ , and the system will be unstable. Thus, this system is stable for the range of loop gain,  $K$ , that ensures that the *open-loop magnitude is less than unity at that frequency where the phase angle is  $180^\circ$  (or, equivalently,  $-180^\circ$ )*. This statement is thus an alternative to the Nyquist criterion for this system.

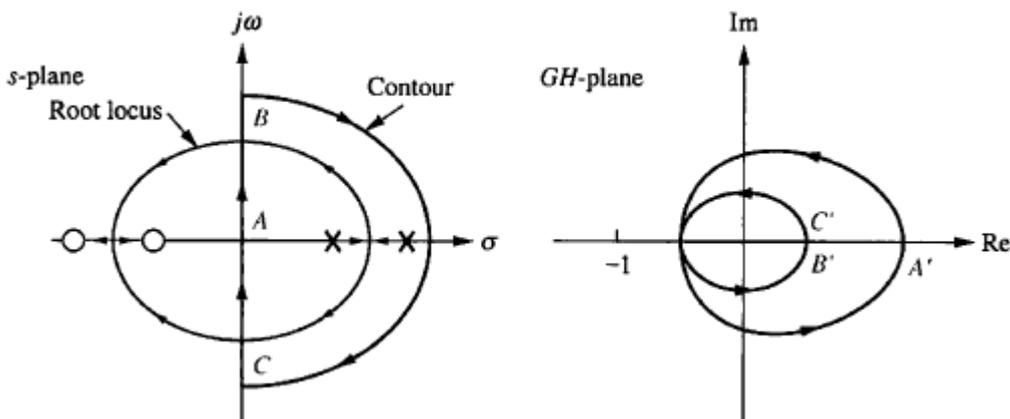


Figure 6. Contour and root locus of system that is unstable for small gain and stable for large gain

Now consider the system shown in above Figure, which is unstable at low values of gain and stable at high values of gain. Since the contour encloses two open-loop poles, two counter clockwise encirclements of the critical point are required for stability. Thus, for this case the system is stable if the *open-loop magnitude is greater than unity at that frequency where the phase angle is  $180^\circ$  (or, equivalently,  $-180^\circ$ )*.

In summary, first determine stability from the Nyquist criterion and the Nyquist diagram. Next interpret the Nyquist criterion and determine whether the mapping of just the positive imaginary axis should have a gain

# Control Systems

of less than or greater than unity at  $180^\circ$ . If the Nyquist diagram crosses  $\pm 180^\circ$  at multiple frequencies, determine the interpretation from the Nyquist criterion.

**PROBLEM:** Find the range of gain for stability and instability, and the gain for marginal stability, for the unity feedback system, where  $G(s) = K/[(s^2 + 2s + 2)(s + 2)]$ . For marginal stability find the radian frequency of oscillation. Use the Nyquist criterion and the mapping of only the positive imaginary axis.

**SOLUTION:** Since the open-loop poles are only in the left-half-plane, the Nyquist criterion tells us that we want no encirclements of -1 for stability. Hence, a gain less than unity at  $\pm 180^\circ$  is required. Begin by letting  $K = 1$  and draw the portion of the contour along the positive imaginary axis as shown in Figure.

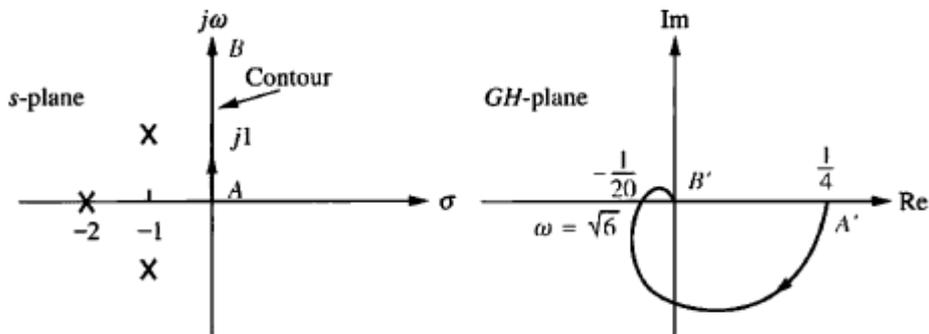


Figure 7. Nyquist diagram of Mapping of positive imaginary axis

In Figure, the intersection with the negative real axis is found by letting  $s = j\omega$  in  $G(s)H(s)$ , setting the imaginary part equal to zero to find the frequency, and then substituting the frequency into the real part of  $G(j\omega)H(j\omega)$ . Thus, for any point on the positive imaginary axis,

$$G(j\omega)H(j\omega) = \frac{1}{(s^2 + 2s + 2)(s + 2)} \Big|_{s=j\omega}$$

$$= 4(1 - \omega^2) - \frac{j\omega(5 - \omega^2)}{16(1 - \omega^2)^2 + \omega^2(5 - \omega^2)^2}$$

Setting the imaginary part equal to zero, we find  $\omega = \sqrt{6}$ . Substituting this value back into equation yields

$$-\left(\frac{1}{20}\right) = \left(\frac{1}{20}\right) \angle 180^\circ.$$

the real part,

This closed-loop system is stable if the magnitude of the frequency response is less than unity at  $180^\circ$ . Hence, the system is stable for  $K < 20$ , unstable for  $K > 20$ , and marginally stable for  $K = 20$ . When the system is marginally stable, the radian frequency of oscillation is  $\sqrt{6}$ .

Example:1

Consider the following transfer function

$$G(s) = \frac{K(s + 1)}{s^2(s + 4)(s + 5)}$$

Putting the value of  $s = j\omega$  in above equation, we obtain

$$G(j\omega) = \frac{k(j\omega + 1)}{(j\omega)^2(j\omega + 4)(j\omega + 5)}$$

The magnitude and phase angle equations:

$$\frac{k(\sqrt{\omega^2 + 1})}{\omega^2(\sqrt{\omega^2 + 16})(\sqrt{\omega^2 + 25})}$$

# Control Systems

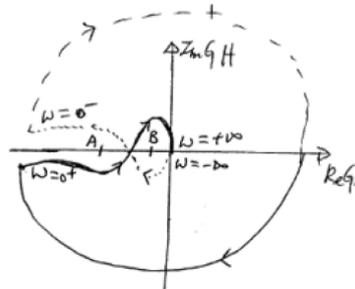
Evaluating magnitude and phase response at  $\omega = 0+$  and  $\omega = +\infty$

At  $\omega = 0+$

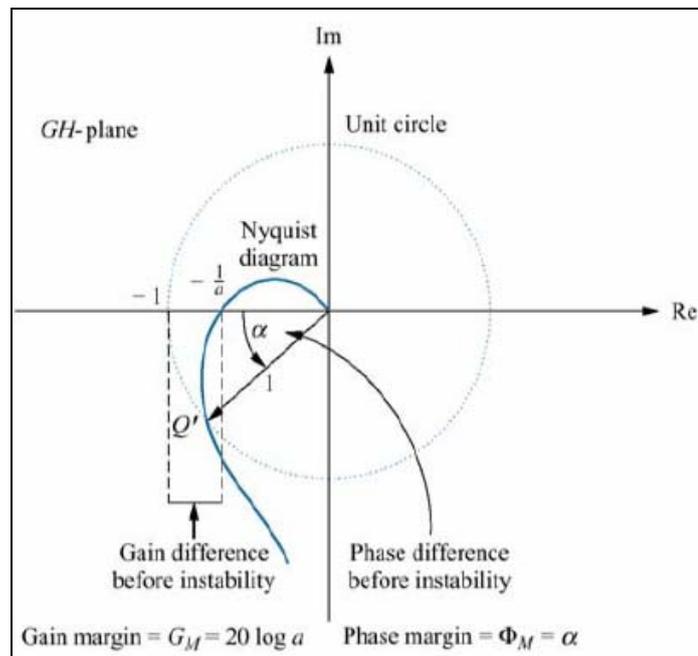
$$|G(j\omega)| \angle G(j\omega) \Rightarrow \infty \angle -180 + \epsilon$$

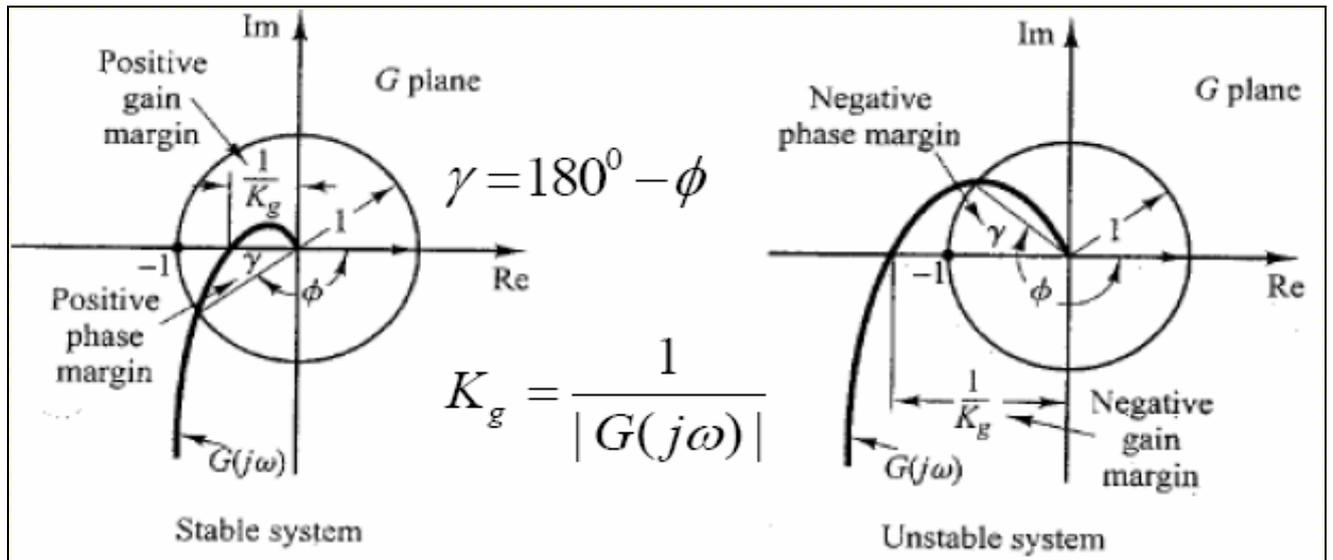
At  $\omega = \infty$

$$|G(j\omega)| \angle G(j\omega) \Rightarrow \infty \angle -270$$



## PHASE AND GAIN MARGIN THROUGH NYQUIST PLOTS:





System is said to be

- Stable, if  $G_M$  and  $\Phi_M$  both are positive, i.e.  $\omega_{pc} > \omega_{gc}$
- Marginally stable, if  $G_M$  and  $\Phi_M$  both are zero i.e.  $\omega_{pc} = \omega_{gc}$
- Unstable, if  $G_M$  and  $\Phi_M$  both are negative i.e.  $\omega_{pc} < \omega_{gc}$

The closed-loop frequency response is the locus of the closed-loop magnitude frequency response for unity feedback system. If the frequency response of an open loop system is plotted in polar coordinates, and superimposed on the top of M-circles, then the closed-loop magnitude frequency response is determined by each intersection of this polar plot with the constant M-circles.

M-circles are contours of constant closed-loop magnitude on Nyquist plane.

Let  $L(j\omega) = x + jy$ . Then  $T(j\omega) = \frac{x + jy}{1 + x + jy}$ . Hence,

$$|T(j\omega)|^2 = M^2 \Rightarrow M^2(1 + x)^2 + M^2y^2 = x^2 + y^2$$

$$\Rightarrow (1 - M^2)x^2 - 2M^2x + (1 - M^2)y^2 = M^2$$

Then two cases are possible:

$M = 1$  then  $x = -\frac{1}{2}$  (vertical line)

$M \neq 1$  then  $(1 - M^2)\left(x^2 - 2\frac{M^2}{1 - M^2}x + \frac{M^4}{(1 - M^2)^2} + y^2\right) = M^2$ , so we get:

$$\left(x - \frac{M^2}{1 - M^2}\right)^2 + y^2 = \left(\frac{M}{1 - M^2}\right)^2$$

$$\left(x + \frac{M^2}{M^2 - 1}\right)^2 + y^2 = \left(\frac{M}{M^2 - 1}\right)^2$$

Constant M-circles are the circles in the complex plane with radius  $M/2\sqrt{|M|^2 - 1}$  centered at  $\left(-\frac{M^2}{M^2 - 1}, 0\right)$

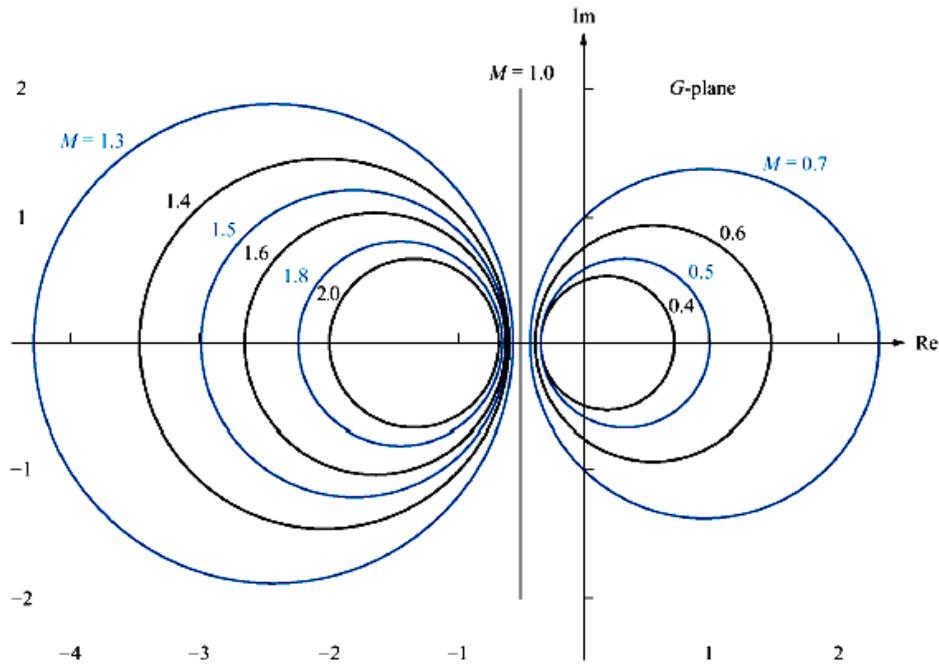


Figure 8. Constant M circles

Constant N-circles

$$\begin{aligned} \angle T(j\omega) = \alpha &= \angle \left( \frac{(x + jy)}{1 + x + jy} \right) \\ &= \tan^{-1} \left( \frac{y}{x} \right) - \tan^{-1} \left( \frac{y}{1 + x} \right) \\ &= \tan^{-1} \left[ \frac{y}{x^2 + x + y^2} \right] \end{aligned}$$

Therefore

$$\tan \alpha = \frac{y}{x^2 + x + y^2} = N$$

For a constant value of  $\alpha$ ,  $N = \tan \alpha$  is also constant.

Rearranging the equation we get,

$$\left(x + \frac{1}{2}\right)^2 + \left(y - \frac{1}{2N}\right)^2 = \frac{N^2 + 1}{(2N)^2}$$

constant N-circles are the circles in the complex plane with radius  $\frac{\sqrt{N^2 + 1}}{2N}$  centered at  $(-1/2, 1/2N)$  (see figure 2). Constant N-circles are the locus of the closed-loop phase frequency response. Similarly to M-circles, if the frequency response of an open loop system is plotted in polar coordinates, and superimposed on the top of N-circles, then the closed-loop phase frequency response is determined by each intersection of this polar plot with the constant N-circles. All the constant N-circles pass through the origin and  $(-1+j0)$  point regardless of the value of N.

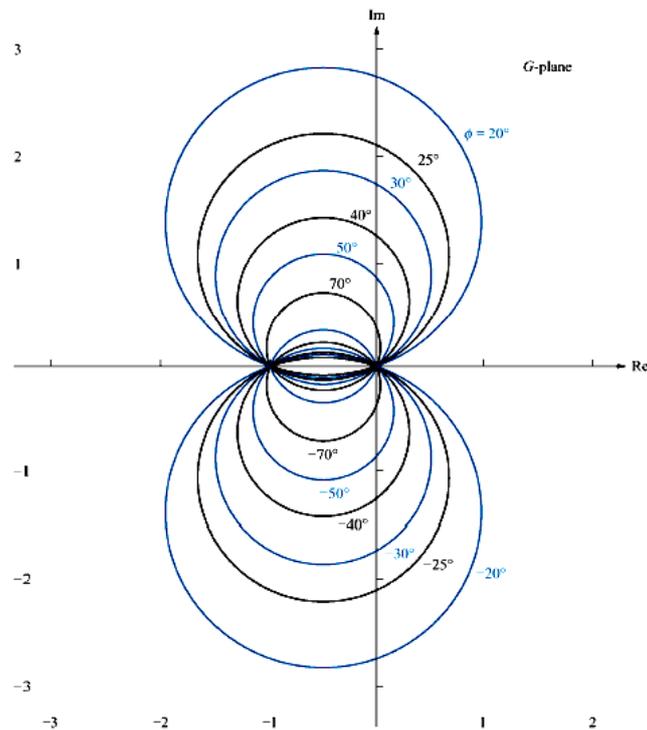


Figure 9. Constant N circles

### Example 1- Closed-loop frequency response from open-loop frequency response

Find the closed-loop frequency response of the unity feedback system with open-loop transfer function

$$G(s) = \frac{50}{s(s+3)(s+6)}$$

using the open-loop polar frequency response curve, constant M-circles, and constant N-circles.

#### Solution

Open-loop frequency response is

$$G(j\omega) = \frac{50}{j\omega(j\omega+3)(j\omega+6)} = \frac{50}{-9\omega^2 + j(18\omega - \omega^3)}$$

Polar plot of  $G(j\omega)$  is shown superimposed over the M- and N-circles in figure 3.

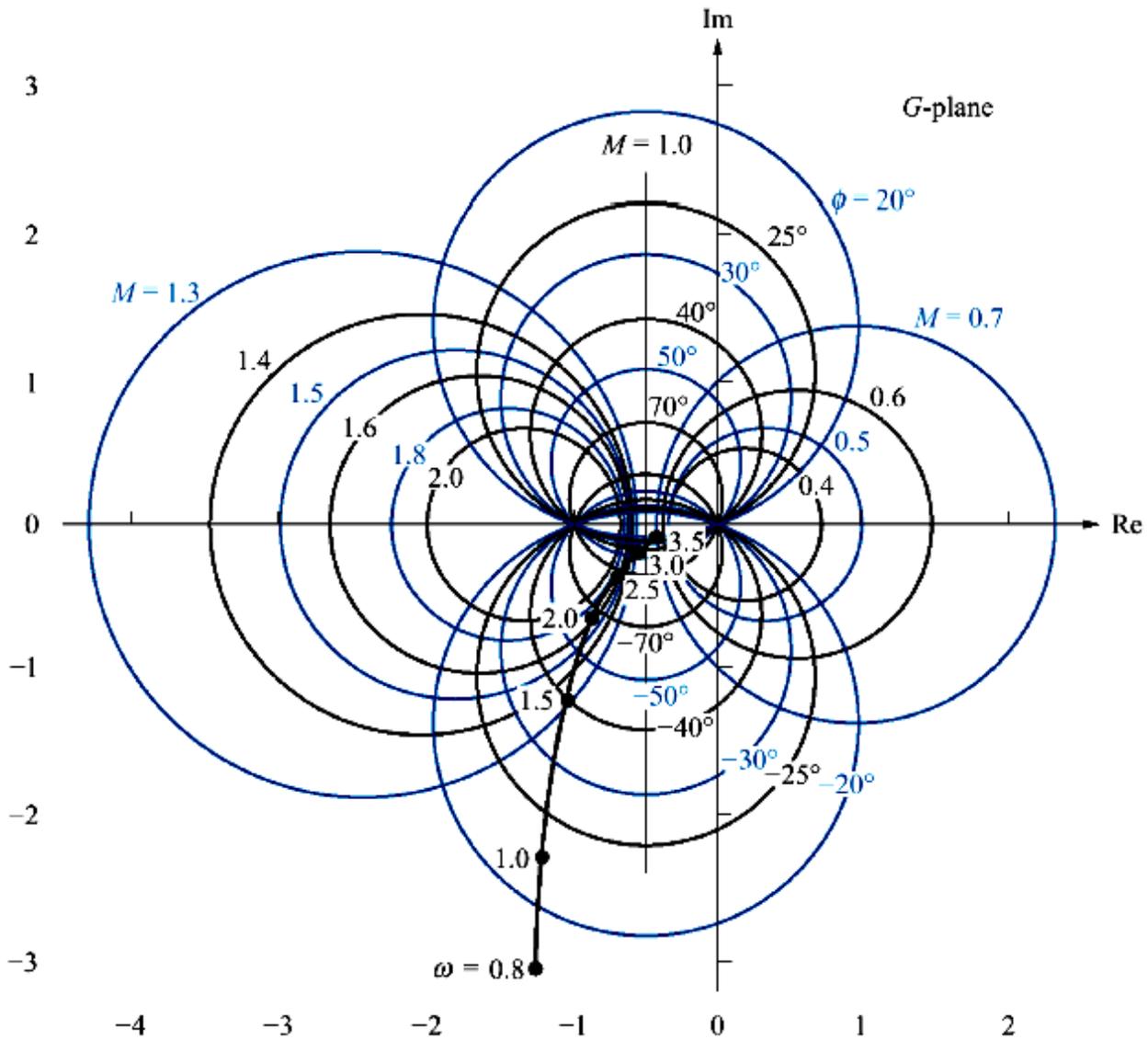


Figure 10. Constant N and M circles

The closed-loop magnitude frequency response can be obtained by finding the intersection of each point of the  $G(j\omega)$  with the M-circles, and the closed-loop phase frequency response can be obtained by finding the intersection of each point of the  $G(j\omega)$  with the N-circles.

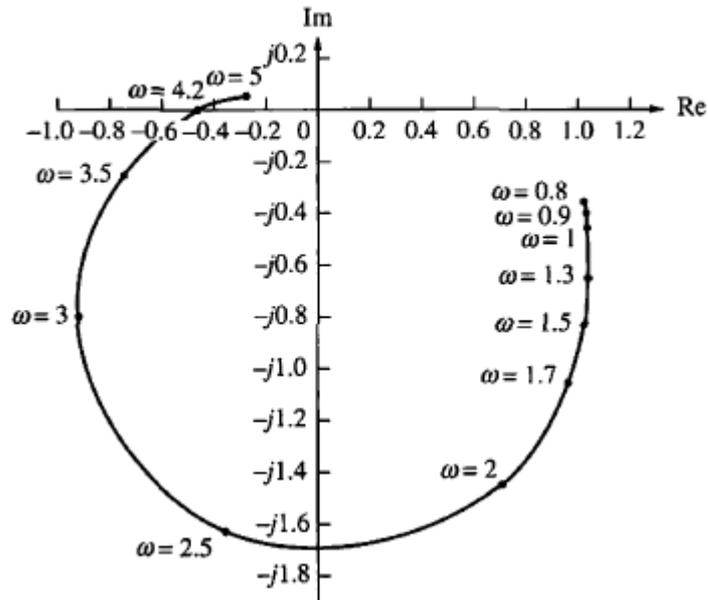
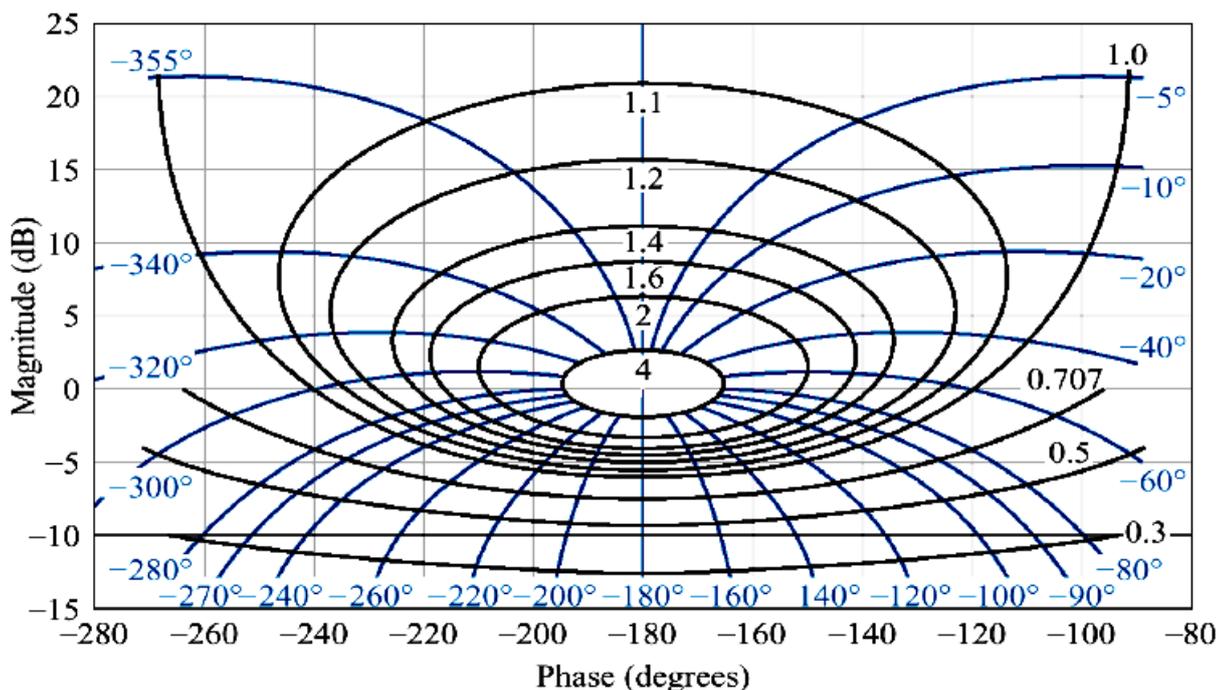


Figure 11. Closed loop frequency response

### Nichols Charts

Since it is easier to construct a bode plot than a polar plot, it is preferable to have constant-M and constant- $\alpha$  contours constructed on logarithmic gain and phase coordinates. N.B. Nichols transformed the constant-M and constant- $\alpha$  contours constructed on logarithmic gain and phase coordinate and the resulting chart is known as the Nichols chart. It displays magnitude response in decibels, so that changes in gain are as simple to handle as in the Bode plot. Nichols chart is a plot of open-loop magnitude in dB vs. open-loop phase. Every point on the constant M- and N-circles is transferred to the Nichols chart (see figure 4). The intersection of the  $G(j\omega)$  with the Nichols chart yields the frequency response of the closed-loop system.



**Example 2-** Closed-loop frequency response from open-loop frequency response using Nichols chart  
 Consider a unity feedback system with the following open-loop transfer function

$$G(s) = \frac{K}{s(s+1)(s+2)}$$

Find the closed-loop frequency response using Nichols chart.

**Solution**

Superimposing the open-loop frequency response for  $K = 1$  on the Nichols chart, we obtain the plot shown in figure 5.

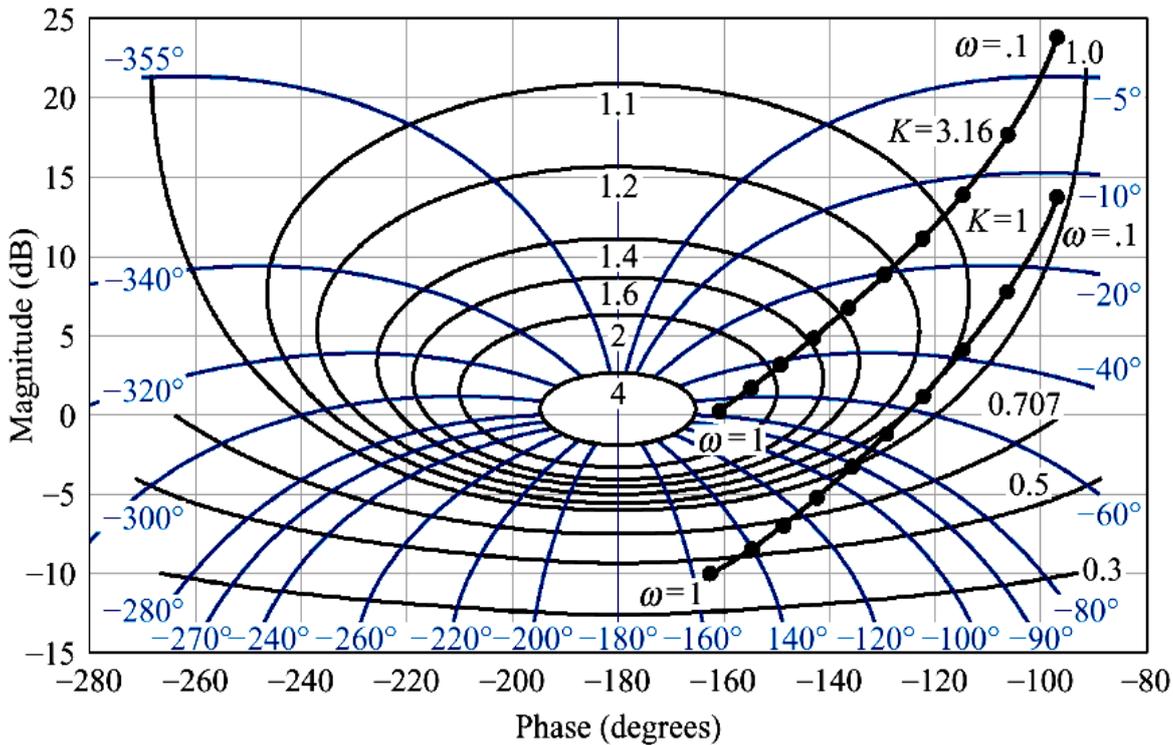
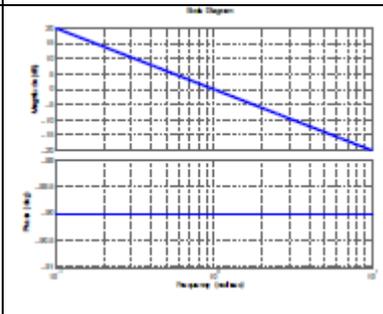
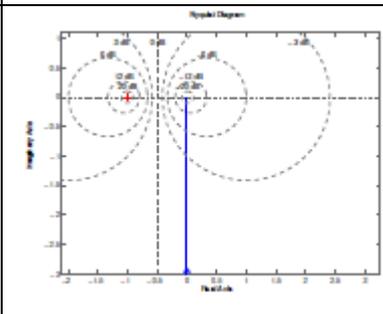
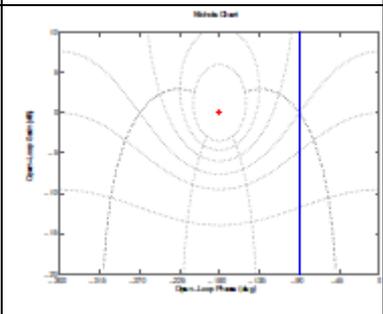
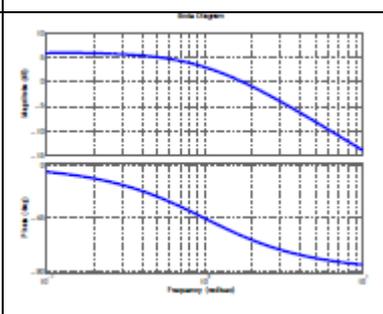
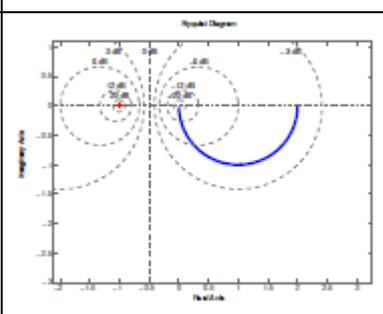
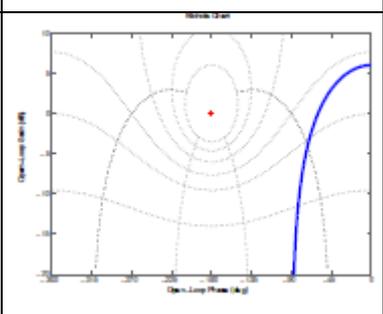
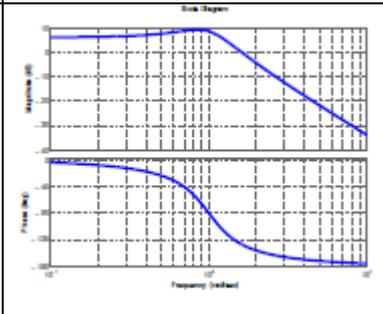
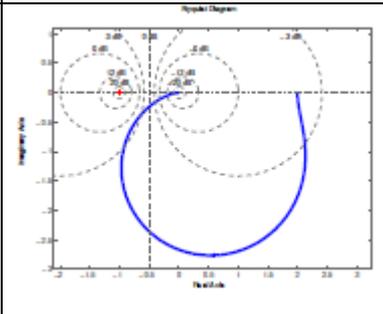
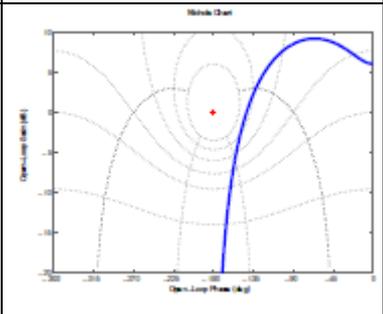
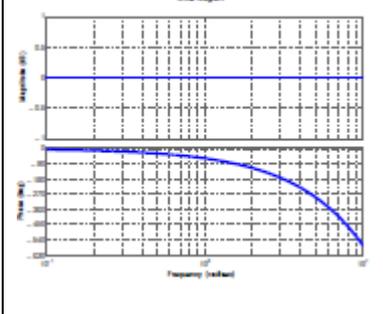
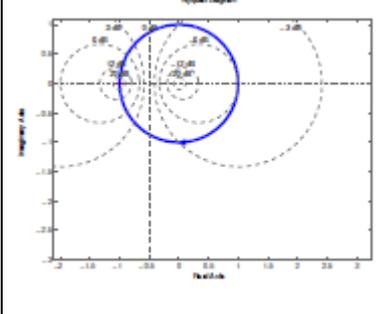
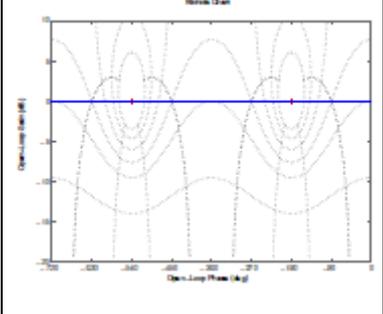


Figure 5: Nichols chart for Example 2

The intersection of the plot of  $G(j\omega)$  with the Nichols chart yields the frequency response of the closed-loop system

If the gain is increased by 10 dB, one should simply raise the curve for  $K = 1$  by 10 dB to obtain the curve for  $K = 3.16$  (10 dB) (see figure 5).

## Nichols Chart for elementary systems

$L(s)$	Bode	Nyquist	Nichols
$\frac{k}{s}$			
$\frac{k}{\tau s + 1}$			
$\frac{k\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$			
$e^{-sh}$			

## 11. Controllers: Concept of Proportional, Derivative and Integral Control actions:

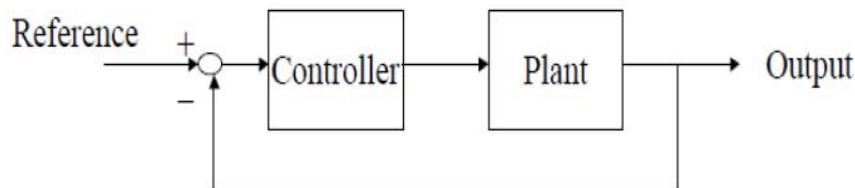
A variety of controls are used to manipulate processes, however the most simple and often most effective is the PID controller. Much more practical than the typical on/off controller, PID controllers allow for much better adjustments to be made in the system. While this is true, there are some advantages to using an on/off controller:

- Relatively simple to design and execute
- Binary sensors and actuators (such as an on/off controller) are generally more reliable and less expensive

Although there are some advantages, there are large disadvantages to using an on/off controller scheme:

- Inefficient (using this control is like driving with full gas and full breaks)
- Can generate noise when seeking stability (can dramatically overshoot or undershoot a set-point)
- Physically wearing on valves and switches (continuously turning valves/switches fully on and fully off causes them to become worn out much quicker)

To allow for much better control and fine-tuning adjustments, most industrial processes use a PID controller scheme.



The controller attempts to correct the error between a measured process variable and desired set-point by calculating the difference and then performing a corrective action to adjust the process accordingly. A PID controller controls a process through three parameters: Proportional (P), Integral (I), and Derivative (D). These parameters can be weighted, or tuned, to adjust their effect on the process. The following section will provide a brief introduction on PID controllers.

The Process Gain (K) is the ratio of change of the output variable (responding variable) to the change of the input variable (forcing function). It specifically defines the sensitivity of the output variable to a given change in the input variable.

$$K = \frac{\Delta \text{Output}}{\Delta \text{Input}}$$

Gain can only be described as a steady state parameter and give no knowledge about the dynamics of the process and is independent of the design and operating variables. A gain has three components that include the sign, the value, and the units. The sign indicates how the output responds to the process input. A positive sign shows that the output variable increases with an increase in the input variable and a negative sign shows that the output variable decreases with an increase in the input variable. The units depend on the process considered that depend on the variables mentioned.

As previously mentioned, controllers vary in the way they correlate the controller input (error) to the controller output (actuating signal). The most commonly used controllers are the proportional- integral-

derivative (PID) controllers. PID controllers relate the error to the actuating signal either in a proportional (P), integral (I), or derivative (D) manner. PID controllers can also relate the error to the actuating signal using a combination of these controls.

## Proportional (P) Control

Proportional control is the simplest form of continuous control that can be used in a closed-looped system. P-only control minimizes the fluctuation in the process variable, but it does not always bring the system to the desired set point. This deviation is known as the offset, and it is usually not desired in a process. The existence of an offset implies that the system could not be maintained at the desired set point at steady state. It is analogous to the systematic error in a calibration curve, where there is always a set, constant error that prevents the line from crossing the origin. The offset can be minimized by combining P-only control with another form of control, such as I- or D- control.

## Mathematical Equations

P-control linearly correlates the controller output (actuating signal) to the error (difference between measured signal and set point). This P-control behavior is mathematically illustrated in Equation 1.

$$c(t) = K_c e(t) + b \quad (1)$$

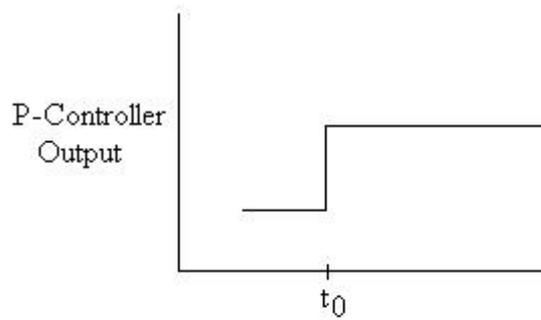
$c(t)$  = controller output

$K_c$  = controller gain

$e(t)$  = error

$b$  = bias

As can be seen from the above equation, P-only control provides a linear relationship between the error of a system and the controller output of the system. Combined with the bias, this algorithm determines the action that the controller should take. A graphical representation of the P-controller output for a step increase in input at time  $t_0$  is shown below in Figure 2. This graph is exactly similar to the step input graph itself.



**Figure 2.** P-controller output for step input.

## Integral (I) Control

Integral control is a second form of feedback control. It is often used because it is able to remove any deviations that may exist. Thus, the system returns to both steady state and its original setting. A negative error will cause the signal to the system to decrease, while a positive error will cause the signal to increase. However, I-only controllers are much slower in their response time than P-only controllers because they are dependent on more parameters. If it is essential to have no offset in the system, then an I-only controller should be used, but it will require a slower response time. This slower response time can be reduced by combining I-only control with another form, such as P or PD control. The philosophy behind the integral control is that deviations will be affected in proportion to the cumulative sum of their magnitude. The key advantage of adding a I-control to your controller is that it will eliminate the offset. The disadvantages are that it can destabilize the controller, and there is an integrator windup, which increases the time it takes for the controller to make changes.

## Mathematical Equations

I-control correlates the controller output to the integral of the error. The integral of the error is taken with respect to time. It is the total error associated over a specified amount of time. This I-control behavior is mathematically illustrated in Equation 2.

$$c(t) = \frac{1}{T_i} \int e(t) dt + c(t_0) \quad (2)$$

$c(t)$  = controller output

$T_i$  = integral time

$$e(t) = \text{error}$$

$$c(t_0) = \text{controller output before integration}$$

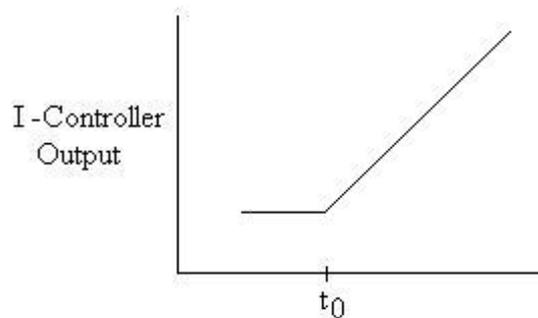
Steady state error for a Step input -

$$\frac{Y(s)}{R(s)} = \frac{G_p(s)}{1 + G_p(s)}$$

$$Y(s) = E(s)G_p(s)$$

$$e_{ss} = \lim_{s \rightarrow 0} e(t) = \lim_{s \rightarrow 0} sE(s) = \lim_{s \rightarrow 0} \frac{sR(s)}{1 + G_p(s)} = \lim_{s \rightarrow 0} \frac{1}{1 + G_p(s)} = \frac{1}{1 + \infty} = 0$$

In this equation, the integral time is the amount of time that it takes for the controller to change its output by a value equal to the error. The controller output before integration is equal to either the initial output at time  $t=0$ , or the controller output at the time one step before the measurement. As expected, this graph represents the area under the step input graph.



**Figure 3.** I-controller output for step input.

## Derivative (D) Control

Unlike P-only and I-only controls, D-control is a form of feed forward control. D-control anticipates the process conditions by analyzing the change in error. It functions to minimize the change of error, thus keeping the system at a consistent setting. The primary benefit of D controllers is to resist change in the system, the most important of these being oscillations. The control output is calculated based on the rate of change of the error with time. The larger the rate of the change in error, the more pronounced the controller response will be.

Unlike proportional and integral controllers, derivative controllers do not guide the system to a steady state. Because of this property, D controllers must be coupled with P, I or PI controllers to properly control the system.

## Mathematical Equations

D-control correlates the controller output to the derivative of the error. The derivative of the error is taken with respect to time. It is the change in error associated with change in time. This D-control behavior is mathematically illustrated in Equation 3.

$$c(t) = T_d \frac{de}{dt} \quad (3)$$

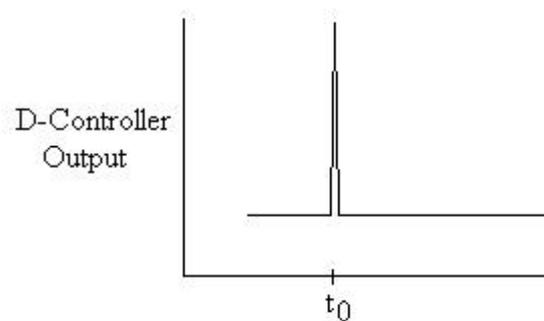
$c(t)$  = controller output

$T_d$  = derivative time constant

$de$  = change in error

$dt$  = change in time

A graphical representation of the D-controller output for a step increase in input at time  $t_0$  is shown below in Figure 4. As expected, this graph represents the derivative of the step input graph.



**Figure 4.** D-controller output for step input.

## Proportional-Integral (PI) Control

One combination is the PI-control, which lacks the D-control of the PID system. PI control is a form of feedback control. It provides a faster response time than I-only control due to the addition of the proportional action. PI control stops the system from fluctuating, and it is also able to return the system to its set point. Although the response time for PI-control is faster than I-only control, it is still up to 50% slower than P-only control. Therefore, in order to increase response time, PI control is often combined with D-only control.

## Mathematical Equations

PI-control correlates the controller output to the error and the integral of the error. This PI-control behavior is mathematically illustrated in Equation 4.

$$c(t) = K_c \left( e(t) + \frac{1}{T_i} \int e(t) dt \right) + C \quad (4)$$

$c(t)$  = controller output

$K_c$  = controller gain

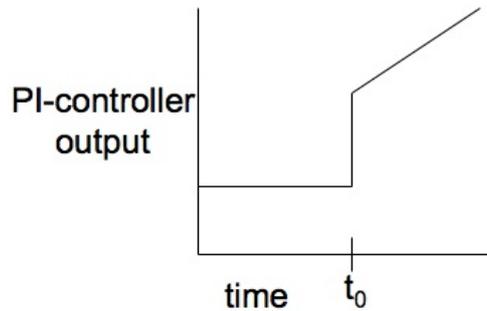
$T_i$  = integral time

$e(t)$  = error

$C$  = initial value of controller

In this equation, the integral time is the time required for the I-only portion of the controller to match the control provided by the P-only part of the controller.

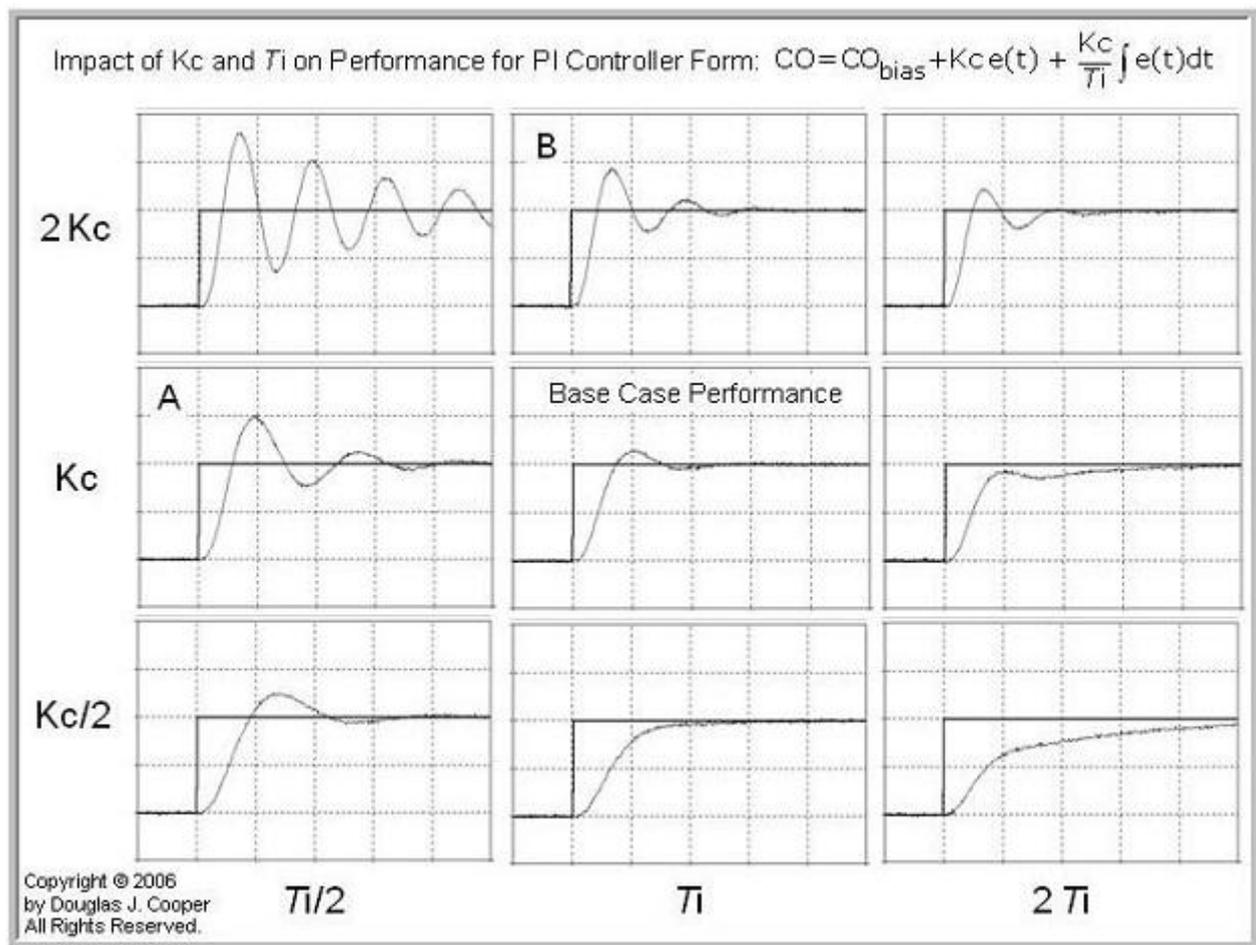
The PI-controller can also be seen as a combination of the P-only and I-only control equations. The bias term in the P-only control is equal to the integral action of the I-only control. The P-only control is only in action when the system is not at the set point. When the system is at the set point, the error is equal to zero, and the first term drops out of the equation. The system is then being controlled only by the I-only portion of the controller. Should the system deviate from the set point again, P-only control will be enacted. A graphical representation of the PI-controller output for a step increase in input at time  $t_0$  is shown below in Figure 5.



**Figure 5.** PI-controller output for step input.

### **Effects of $K_c$ and $T_i$**

With a PI control system, controller activity (aggressiveness) increases as  $K_c$  and  $T_i$  decreases, however they can act individually on the aggressiveness of a controller's response. Consider Figure 6 below with the center graph being a linear second order system base case.



**Figure 6.** Effects of  $K_c$  and  $T_i$  [2]

The plot depicts how  $T_i$  and  $K_c$  both affect the performance of a system, whether they are both affecting it or each one is independently doing so. Regardless of integral time, increasing controller gain (moving from bottom to top on the plot) will increase controller activity. Similarly, decreasing integral time (moving right to left on the plot) will increase controller activity independent of controller gain. As expected, increasing  $K_c$  and decreasing  $T_i$  would compound sensitivity and create the most aggressive controller scenario.

Another noteworthy observation is the plot with a normal  $K_c$  and double  $T_i$ . The plot depicts how the proportional term is practical but the integral is not receiving enough weight initially, causing the slight oscillation before the integral term can finally catch up and help the system towards the set point.

## Proportional-Derivative (PD) Control

Another combination of controls is the PD-control, which lacks the I-control of the PID system. PD-control is combination of feed-forward and feedback control, because it operates on both the current process conditions and predicted process conditions. In PD-control, the control output is a linear combination of the error signal and its derivative. PD-control contains the proportional control's damping of the fluctuation and the derivative control's prediction of process error.

## Mathematical Equations

As mentioned, PD-control correlates the controller output to the error and the derivative of the error. This PD-control behavior is mathematically illustrated in Equation 5.

$$c(t) = K_c \left( e(t) + T_d \frac{de}{dt} \right) + C \quad (5)$$

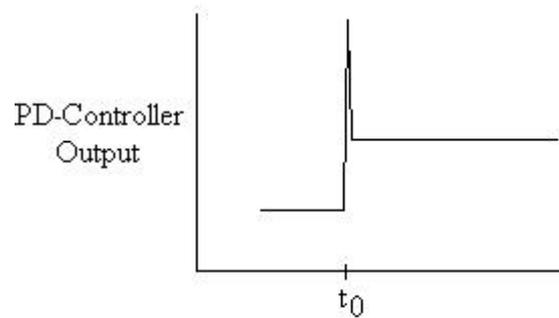
$c(t)$  = controller output

$K_c$  = proportional gain

$e$  = error

$C$  = initial value of controller

The equation indicates that the PD-controller operates like a simplified PID-controller with a zero integral term. Alternatively, the PD-controller can also be seen as a combination of the P-only and D-only control equations. In this control, the purpose of the D-only control is to predict the error in order to increase stability of the closed loop system. P-D control is not commonly used because of the lack of the integral term. Without the integral term, the error in steady state operation is not minimized. P-D control is usually used in batch pH control loops, where error in steady state operation does not need to be minimized. In this application, the error is related to the actuating signal both through the proportional and derivative term. A graphical representation of the PD-controller output for a step increase in input at time  $t_0$  is shown below in Figure 6. Again, this graph is a combination of the P-only and D-only graphs, as expected.



**Figure 12.** PD-controller output for step input.

## Proportional-Integral-Derivative (PID) Control

Proportional-integral-derivative control is a combination of all three types of control methods. PID-control is most commonly used because it combines the advantages of each type of control. This includes a quicker response time because of the P-only control, along with the decreased/zero offset from the combined derivative and integral controllers. This offset was removed by additionally using the I-control. The addition of D-control greatly increases the controller's response when used in combination because it predicts disturbances to the system by measuring the change in error. On the contrary, as mentioned previously, when used individually, it has a slower response time compared to the quicker P-only control. However, although the PID controller seems to be the most adequate controller, it is also the most expensive controller. Therefore, it is not used unless the process requires the accuracy and stability provided by the PID controller.

## Mathematical Equations

PID-control correlates the controller output to the error, integral of the error, and derivative of the error. This PID-control behavior is mathematically illustrated in Equation 6.

$$c(t) = K_c \left( e(t) + \frac{1}{T_i} \int e(t) dt + T_d \frac{de}{dt} \right) + C \quad (6)$$

$$c(t) = \text{controller output}$$

$$K_c = \text{controller gain}$$

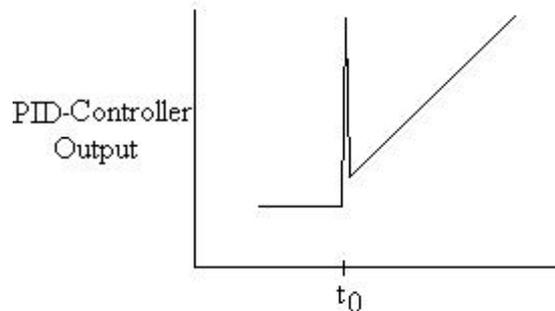
$$e(t) = \text{error}$$

$$T_i = \text{integral time}$$

$$T_d = \text{derivative time constant}$$

$$C = \text{initial value of controller}$$

As shown in the above equation, PID control is the combination of all three types of control. In this equation, the gain is multiplied with the integral and derivative terms, along with the proportional term, because in PID combination control, the gain affects the I and D actions as well. Because of the use of derivative control, PID control cannot be used in processes where there is a lot of noise, since the noise would interfere with the predictive, feed-forward aspect. A graphical representation of the PID-controller output for a step increase in input at time  $t_0$  is shown below in Figure 7. This graph resembles the qualitative combination of the P-only, I-only, and D-only graphs.



**Figure 7.** PID-controller output for step input.

In addition to PID-control, the P-, I-, and D- controls can be combined in other ways. These alternative combinations are simplifications of the PID-control.

## Summary Tables

A summary of the advantages and disadvantages of the three controls is shown below in Table 1.

**Table 1.** Advantages and disadvantages of controls

	Proportional (P)	Integral (I)	Derivative (D)
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# Control Systems

<b>Advantages</b>	Fast response time Minimizes fluctuation	Reduces steady state error.	Improves stability Controls process with rapidly changing outputs
<b>Disadvantages</b>	Contains large offset Does not bring system to desired set point	Slow response time Reduces stability margins	Highly sensitive to noise Requires combined use with another controller

## Effects of Coefficients:

Parameter	Speed of Response	Stability	Accuracy
Increasing K	Increases	Deteriorate	Improves
Increasing $K_i$	Decreases	Deteriorate	Improves
Increasing $K_d$	Increases	Improves	No impact

## What is tuning?

Tuning is adjustment of control parameters to the optimum values for the desired control response. Stability is a basic requirement. However, different systems have different behavior, different applications have different requirements, and requirements may conflict with one another.

### Ziegler–Nichols tuning method:

This method was introduced by John G. Ziegler and Nathaniel B. Nichols in the 1940s. The Ziegler-Nichols' closed loop method is based on experiments executed on an established control loop (a real system or a simulated system).

### Closed Loop (Feedback Loop)

1. Remove integral and derivative action. Set integral time ( $T_i$ ) to 999 or its largest value and set the derivative controller ( $T_d$ ) to zero.
2. Create a small disturbance in the loop by changing the set point. Adjust the proportional, increasing and/or decreasing, the gain until the oscillations have constant amplitude.
3. Record the gain value ( $K_u$ ) and period of oscillation ( $P_u$ ).
4. Plug these values into the Ziegler-Nichols closed loop equations and determine the necessary settings for the controller.

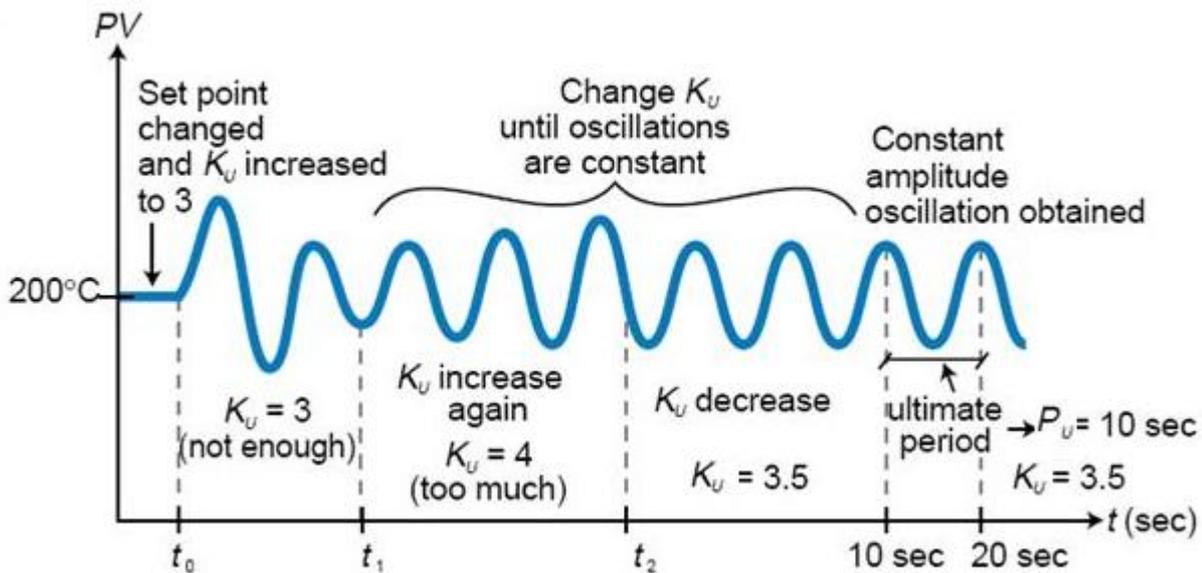


Figure 1. System tuned using the Ziegler-Nichols closed-loop tuning method

	$K_c$	$T_i$	$T_D$
P	$\frac{K_v}{2}$		
PI	$\frac{K_v}{2.2}$	$\frac{P_u}{1.2}$	
PID	$\frac{K_v}{1.7}$	$\frac{P_u}{2}$	$\frac{P_u}{8}$

### Advantages

1. Easy experiment; only need to change the P controller
2. Includes dynamics of whole process, which gives a more accurate picture of how the system is behaving

### Disadvantages

1. Experiment can be time consuming
2. Can venture into unstable regions while testing the P controller, which could cause the system to become out of control

## Ziegler-Nichols Open-Loop Tuning Method or Process Reaction Method -

This method remains a popular technique for tuning controllers that use proportional, integral, and derivative actions. The Ziegler-Nichols open-loop method is also referred to as a process reaction method, because it tests the open-loop reaction of the process to a change in the control variable output. This basic test requires that the response of the system be recorded, preferably by a plotter or computer. Once certain process response values are found, they can be plugged into the Ziegler-Nichols equation with specific multiplier constants for the gains of a controller with either P, PI, or PID actions.

In this method, the variables being measured are those of a system that is already in place. A disturbance is introduced into the system and data can then be obtained from this curve. First the system is allowed to reach steady state, and then a disturbance,  $X_o$ , is introduced to it. The percentage of disturbance to the system can be introduced by a change in either the set point or process variable. For example, if you have a thermometer in which you can only turn it up or down by 10 degrees, then raising the temperature by 1 degree would be a 10% disturbance to the system. These types of curves are obtained in open loop systems when there is no control of the system, allowing the disturbance to be recorded. The process reaction curve method usually produces a response to a step function change for which several parameters may be measured which include: transportation lag or dead time,  $\tau_{dead}$ , the time for the response to change,  $\tau$ , and the ultimate value that the response reaches at steady-state,  $M_u$ .

$\tau_{dead}$  = transportation lag or dead time: the time taken from the moment the disturbance was introduced to the first sign of change in the output signal

$\tau$  = the time for the response to occur

$X_o$  = the size of the step change

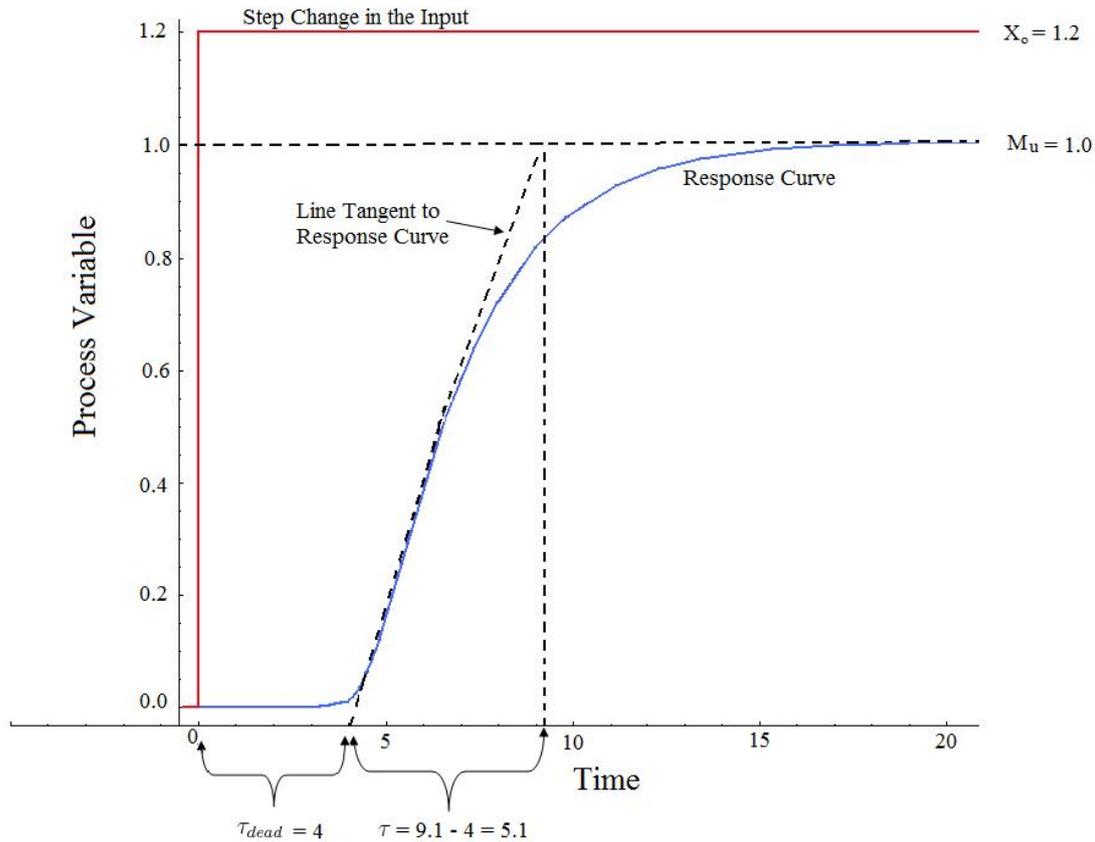
$M_u$  = the value that the response goes to as the system returns to steady-state

$$R = \frac{\tau_{dead}}{\tau}$$

$$K_0 = \frac{X_o \tau}{M_u \tau_{dead}}$$

An example for determining these parameters for a typical process response curve to a step change is shown below.

In order to find the values for  $\tau_{dead}$  and  $\tau$ , a line is drawn at the point of inflection that is tangent to the response curve and then these values are found from the graph.



To use the Ziegler-Nichols open-loop tuning method, you must perform the following steps:

1. Make an open loop step test
2. From the process reaction curve determine the transportation lag or dead time,  $\tau_{dead}$ , the time constant or time for the response to change,  $\tau$ , and the ultimate value that the response reaches at steady-state,  $M_u$ , for a step change of  $X_o$ .

$$K_0 = \frac{X_o \tau}{M_u \tau_{dead}}$$

3. Determine the loop tuning constants. Plug in the reaction rate and lag time values to the Ziegler-Nichols open-loop tuning equations for the appropriate controller—P, PI, or PID—to calculate the controller constants. Use the table below.

Table 2. Open-Loop Calculations of  $K_c$ ,  $T_i$ ,  $T_d$

	$K_c$	$T_i$	$T_D$
P	$K_0$		
PI	$0.9 K_0$	$3.3 \tau_{dead}$	
PID	$1.2 K_0$	$2 \tau_{dead}$	$0.5 \tau_{dead}$

## Advantages

1. Quick and easier to use than other methods
2. It is a robust and popular method
3. Of these two techniques, the Process Reaction Method is the easiest and least disruptive to implement

## Disadvantages

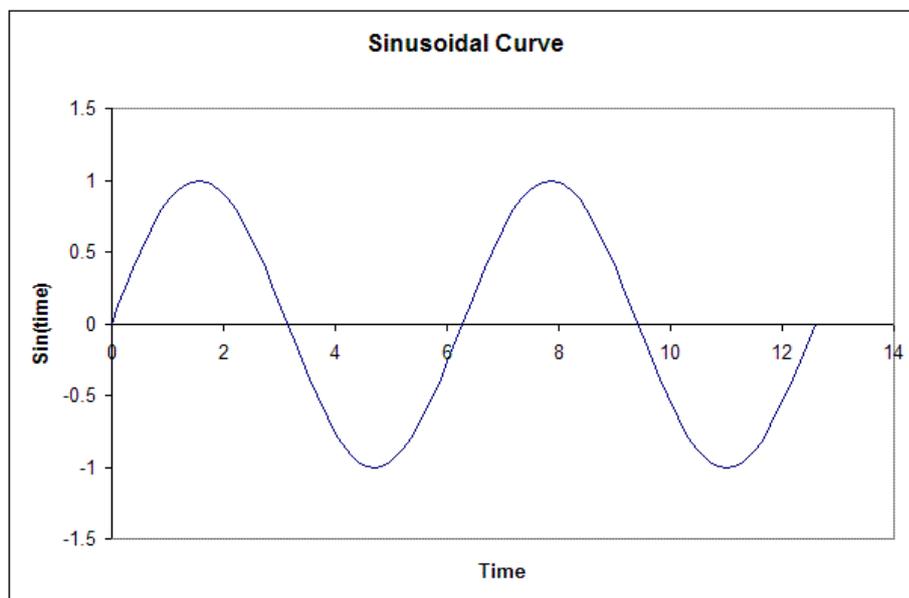
1. It depends upon purely proportional measurement to estimate I and D controllers.
2. Approximations for the  $K_c$ ,  $T_i$ , and  $T_d$  values might not be entirely accurate for different systems.
3. It does not hold for I, D and PD controllers

## Example 1

### Problem

You're a controls engineer working for Flawless Design company when your optimal controller breaks down. As a backup, you figure that by using coarse knowledge of a classical method, you may be able to sustain development of the product. After adjusting the gain to one set of data taken from a controller, you find that your ultimate gain is 4.3289.

From the adjusted plot below, determine the type of loop this graph represents; then, please calculate  $K_c$ ,  $T_i$ , and  $T_d$  for all three types of controllers.

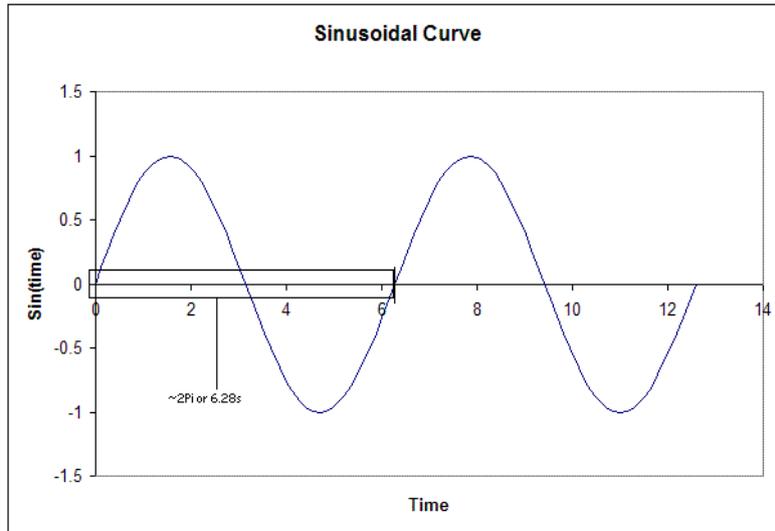


### Solution

From the fact that this graph oscillates and is not a step function, we see that this is a closed loop. Thus, the values will be calculated accordingly.

# Control Systems

We're given the Ultimate gain,  $K_u = 4.3289$ . From the graph below, we see that the ultimate period at this gain is  $P_u = 6.28$



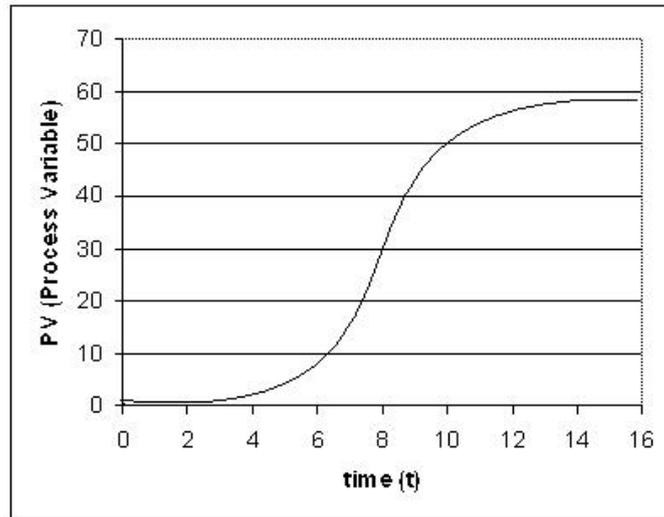
From this, we can calculate the  $K_c$ ,  $T_i$ , and  $T_d$  for all three types of controllers. The results are tabulated below. (Results were calculated from the Ziegler-Nichols closed-loop equations.)

			$K_c$	$T_i$	$T_D$
$K_u$	4.3289	P	2.1645		
$P_u$	6.28	PI	1.9677	5.2333	
		PID	2.5464	3.14	0.785

## Example 2

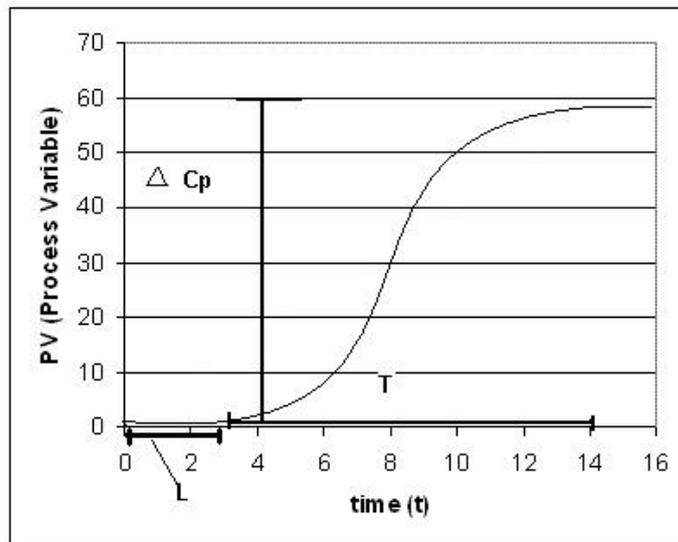
### Problem

Your partner finds another set of data after the controller breaks down and decides to use the Cohen-Coon method because of the slow response time for the system. They also noticed that the control dial, which goes from 0-8, was set at 3 instead of 1. Luckily the response curve was obtained earlier and is illustrated below. From this data he wanted to calculate  $K_c$ ,  $T_i$  and  $T_d$ . Help him to determine these values. Note that the y-axis is percent change in the process variable.



## Solution

In order to solve for  $K_c$ ,  $T_i$  and  $T_d$ , you must first determine  $L$ ,  $\Delta C_p$ , and  $T$ . All of these values may be calculated by the reaction curve given.



From the process reaction curve we can find that:

$$L = 3$$

$$T = 11$$

$$\Delta C_p = 0.55 \text{ (55\%)}$$

Now that these three values have been found  $N$  and  $R$  may be calculated using the equations below.

$$N = \frac{\Delta C_p}{T}$$

$$R = \frac{L}{T} = \frac{NL}{\Delta C_p}$$

# Control Systems

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Using these equations you find that

$$N = .05$$

$$R = 0.27$$

We also know that since the controller was moved from 1 to 3, so a 200% change.

$$P = 2.00$$

We use these values to calculate  $K_c$ ,  $T_i$ , and  $T_d$ , for the three types of controllers

	$K_c$	$T_i$	$T_d$
P	14.53		
PI	12.3	6.42	
PID	18.68	6.65	1.04