ADVANCE CONTROL SYSTEM ENGINEERING

ADVANCED CONTROL SYSTEMS (PEEC5414)

Module-I : (15 Hours) Discrete - Time Control Systems :

Introduction: Discrete Time Control Systems and Continuous Time Control Systems, Sampling Process.

Digital Control Systems: Sample and Hold, Analog to digital conversion, Digital to analog conversion.

The Z-transform: Discrete-Time Signals, The Z-transform, Z-transform of Elementary functions, Important properties and Theorms of the Z-transform. The inverse Ztransform, Z-Transform method for solving Difference Equations.

Z-Plane Analysis of Discrete Time Control Systems: Impulse sampling & Data Hold, Reconstruction of Original signals from sampled signals: Sampling theorm, folding, aliasing. Pulse Transfer function: Starred Laplace Transform of the signal involving Both ordinary and starred Laplace Transforms; General procedures for obtaining pulse Transfer functions, Pulse Transfer function of open loop and closed loop systems.

Mapping between the s-plane and the z-plane, Stability analysis of closed loop systems in the z-plane: Stability analysis by use of the Bilinear Transformation and Routh stability critgion, Jury stability. **Test. Book No.** 1: 1.1; 1.2; 1.4; 2.1; 2.2; 2.3; 2.4; 2.5; 2.6; 3.2; 3.4; 3.5; 4.2; 4.3.

Module -II : (15 Hours) State Variable Analysis & Design:

Introduction: Concepts of State, State Variables and State Model (of continuous time systems): State Model of Linear Systems, State Model for Single-Input-Single-Output Linear Systems, Linearization of the State Equation. **State Models for Linear Continuous – Time Systems:** State-Space Representation Using Physical Variables, State – space Representation Using Phase Variables, Phase variable formulations for transfer function with poles and zeros, State – space Representation using Canonical Variables, Derivation of Transfer Function for State Model. **Diagonalization:** Eigenvalues and Eigenvectors, Generalized Eigenvectors.

Solution of State Equations: Properties of the State Transition Matrix, Computation of State Transition Matrix, Computation by Techniques Based on the Cayley-Hamilton Theorem, Sylvester's Expansion theorm. **Concepts of Controllability and Observability:** Controllability, Observability, Effect of Pole-zero Cancellation in Transfer Function. **Pole Placement by State Feedback, Observer Systems. State Variables and Linear Discrete – Time Systems:** State Models from Linear Difference Equations/z-transfer Functions, Solution of State Equations (Discrete Case), An Efficient Method of Discretization and Solution, Linear Transformation of State Vector (Discrete-Time Case), Derivation of z-Transfer Function from Discrete-Time State Model. **Book No.** 2: 12.1 to 12.9.

Module -III : (12 Hours) Nonlinear Systems :

Introduction : Behaviour of Non linear Systems, Investigation of nonlinear systems.

Common Physical Non Linearities: Saturation, Friction, Backlash, Relay, Multivariable Nonlinearity.

The Phase Plane Method: Basic Concepts, Singular Points: Nodal Point, Saddle Point, Focus Point, Centre or Vortex Point, **Stability of Non Linear Systems:** Limit Cycles, **Construction of Phase**

Trajectories: Construction by Analytical Method, Construction by Graphical Methods. **The Describing Function Method: Basic Concepts: Derivation of Describing Functions:** Dead-zone and Saturation, Relay with Dead-zone and Hysteresis, Backlash. **Stability Analysis by Describing Function Method:** Relay with Dead Zone, Relay with Hysteresis, Stability Analysis by Gain-phase Plots. **Jump Resonance. Liapunov's Stability Analysis: Introduction, Liapunov's Stability Critrion:** Basic Stability Theorems, Liapunov Functions, Instability. **Direct Method of Liapunov & the Linear System:** Methods of constructing Liapunov functions for Non linear Systems.
Book No. 2: 13.1 to 13.4; 15.1 to 15.10.

Text :

1. Discrete-Time Control System, by K.Ogata, 2nd edition (2009), PHI.
2. Control Systems Engineering, by I.J. Nagrath and M.Gopal., 5th Edition (2007 / 2009), New Age International (P) Ltd. Publishers.

Reference :

1. Design of Feedback Control Systems by Stefani, Shahian, Savant, Hostetter, Fourth Edition (2009), Oxford University Press.
2. Modern Control Systems by K.Ogata, 5th Edition (2010), PHI.
3. Modern Control Systems by Richard C. Dorf. And Robert, H.Bishop, 11th Edition (2008), Pearson Education Inc. Publication.
4. Control Systems (Principles & Design) by M.Gopal, 3rd Edition (2008), Tata Mc.Graw Hill Publishing Company Ltd.
5. Control Systems Engineering by Norman S.Nise, 4th Edition (2008), Wiley India (P) Ltd.

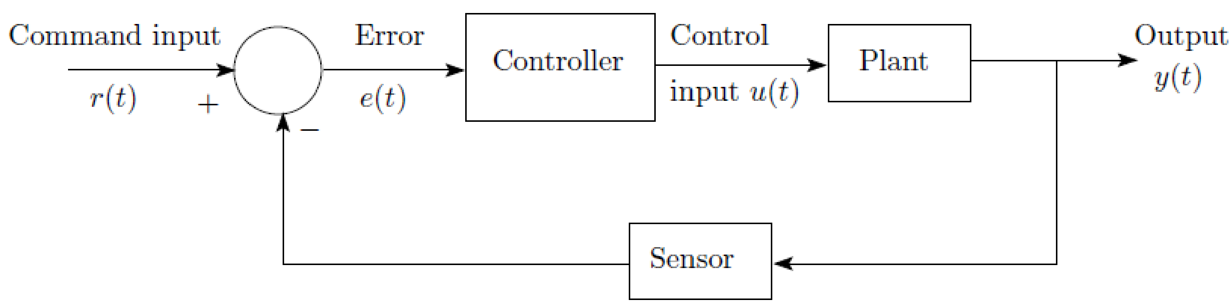
Module 1: Discrete-Time Control systems

Lecture Note 1 (Introduction)

Continuous time Control System:

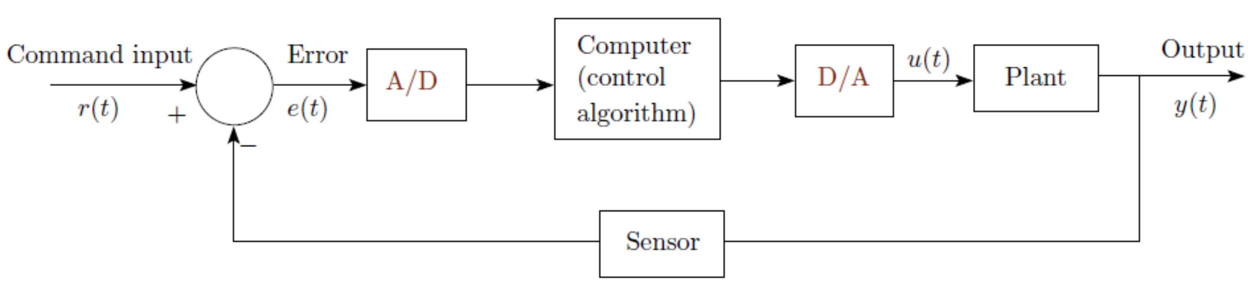
In continuous time control systems, all the system variables are continuous signals. Whether the system is linear or nonlinear, all variables are continuously present and therefore known (available) at all times. A typical continuous time control system is shown in Figure below.

(Closed loop continuous-time control system)



Discrete time Control System:

Discrete time control systems are control systems in which one or more variables can change only at discrete instants of time. These instants, which may be denoted by kT(k=0,1,2,…) specify the times at which some physical measurement is performed or the times at which the memory of a digital computer is read out.



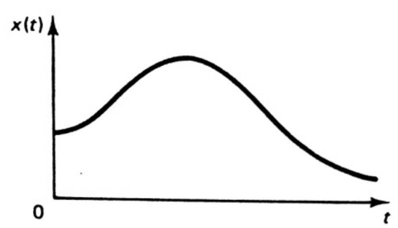
(Block diagram of a discrete-time control system)

Continuous time control systems whose signals are continuous in time are described by differential equation, whereas discrete control systems that involve sampled data signals or digital signals and possibly continuous time signals as well are described by difference equation.

Sampling Process:

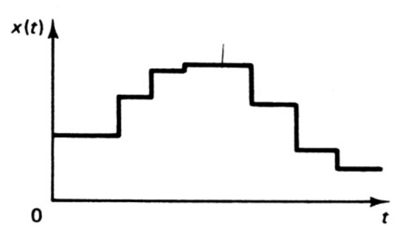
Sampling is a process by which a continuous time signal is converted into a sequence of numbers at discrete time intervals. A sampling process is used whenever a control system involves a digital computer. Also a sampling process occurs whenever measurements necessary for control are obtained in an intermittent fashion. The sampling process is usually followed by a quantization process in which the sampled analog amplitude is replaced by a digital amplitude (represented by a binary number).Then the digital signal is processed by the computer and the output is sampled and fed to a hold circuit. The output of the hold circuit is a continuous time signal and is fed to the actuator for the control of the plant.

(a)



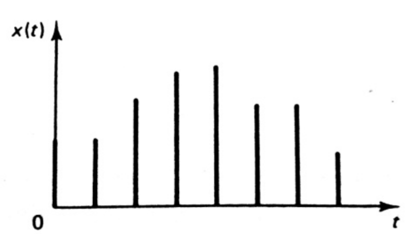
(Continuous-time analog signal)

(b)



(Continuous-time quantized signal)

(c)



(Sampled-data signal)

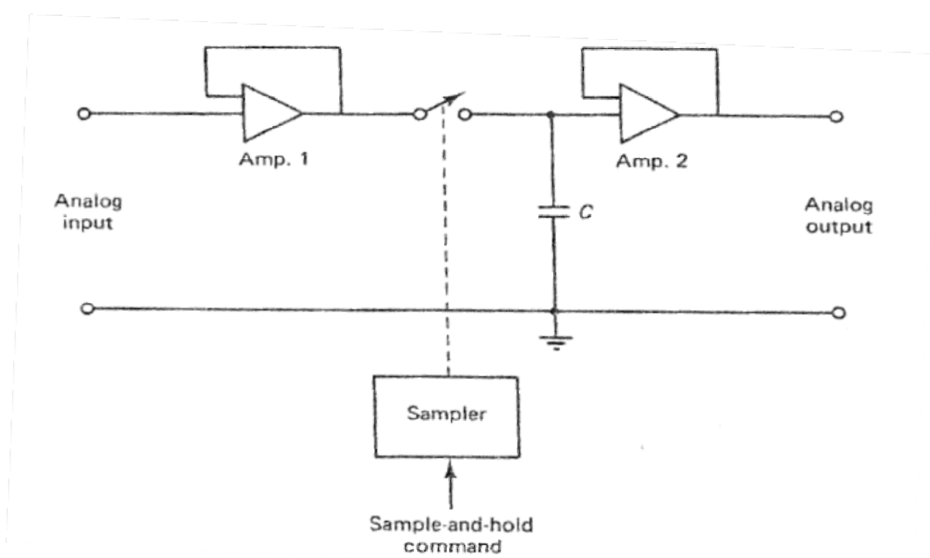
Lecture Note 2 (Digital Control Systems)

Sample and Hold Circuits:

A sampler in a digital system converts an analog signal into a train of amplitude-modulated pulses. The hold circuit holds the value of the sampled pulse signal over a specified period of time. The sample and hold is necessary in the A/D converter to produce a number that accurately represents the input signal at the sampling instant.

The sample and hold circuit is an analog circuit in which an input voltage is acquired and then stored on a high quality capacitor as shown in figure below. Op-Amp 1 is an input buffer amplifier with a high input impedance and Op-Amp 2 is the output amplifier that buffers the voltage on the hold capacitor.

The two modes of operation for a sample and hold circuit are tracking mode & hold mode. When the switch is closed the operating mode is the tracking mode in which the charge on the capacitor in the circuit tracks the input voltage. When the switch is open the operating

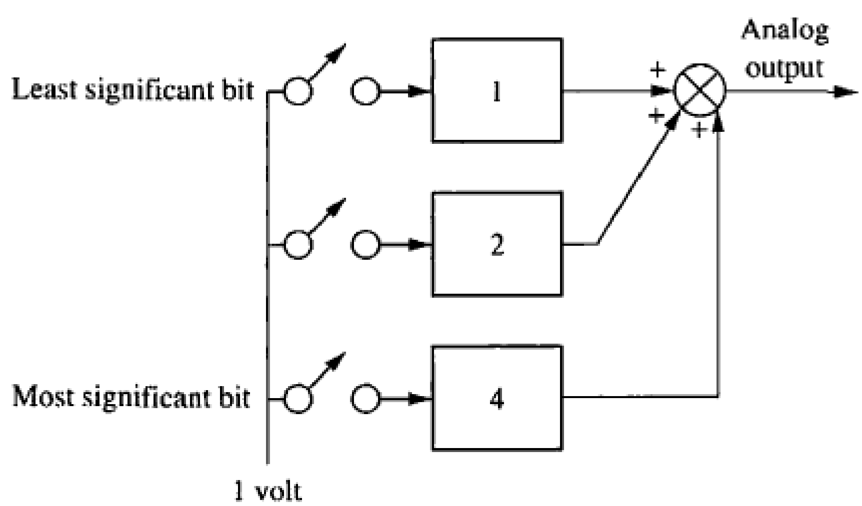


mode is the hold mode in which the capacitor voltage holds constant for a specified time period.

(sample and hold circuit)

Digital to Analog Conversion:

Digital-to-analog conversion is simple and effectively instantaneous. Properly weighted voltages are summed together to yield the analog output. For example, in Figure below, three weighted voltages are summed. The three-bit binary code is represented by the switches. Thus, if the binary number is 110_2, the center and bottom switches are on, and the analog output is 6 volts. In actual use, the switches are electronic and are set by the input binary code.

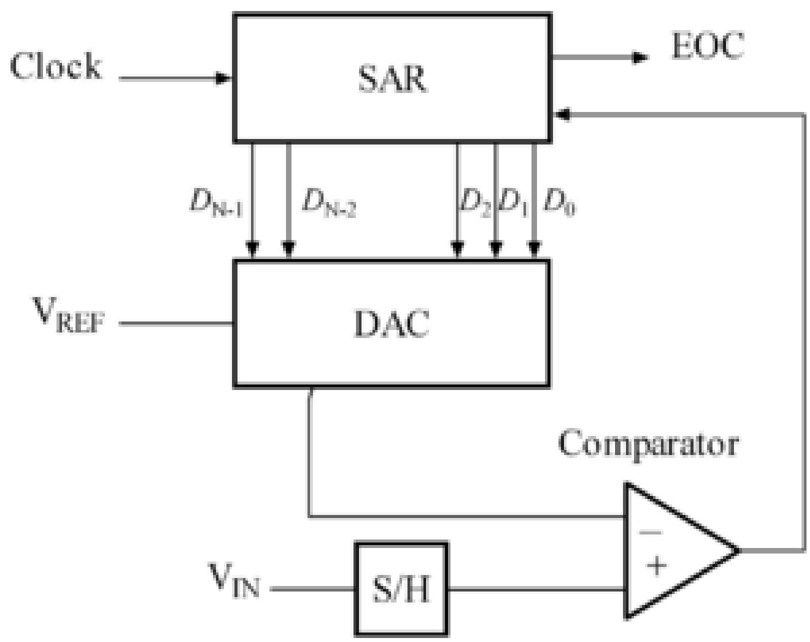


(Digital to Analog converter)

Analog to Digital Conversion:

The process by which a sampled analog signal is quantized and converted to a binary number is called analog to digital conversion. The A/D converter performs the operation of sample-and –hold, quantizing and encoding.

The simplest type of A/D converter is the counter type. The basic principle on which it works is the clock pulses are applied to the digital counter in such a way that the output voltage of the D/A converter (that is part of the feedback loop in the A/D converter) is stepped up one least significant bit at a time, and the output voltage is compared with the analog input



voltage once for each pulse. When the output voltage has reached the magnitude of the input voltage, the clock pulses are stopped. The counter output voltage is then the digital output.

(Counter type analog to digital converter)

Lecture Note 3(The Z-Transform)

Z-transform:

Z-transform is a mathematical tool commonly used for the analysis and synthesis of discrete time control systems. The role of Z-transform in discrete time systems is similar to that of the Laplace transform in continuous systems. In considering Z-transform of a time function x(t), we consider only the sampled values of x(t), i.e.,x(0), x(T), x(2T)....... where T is the sampling period.

$$X(z) = Z[x(t)] = Z[x(kT)] = \sum_{k=0}^{\infty} x(kT)z^{-k}$$

For a sequence of numbers x(k)

$$X(z) = Z[x(k)] = \sum_{k=0}^{\infty} x(k)z^{-k}$$

The above transforms are referred to as one sided z-transform. In one sided z-ransform, we assume that x(t) = 0 for t < 0 or x(k) = 0 for k < 0. In two sided z-transform, we assume that −1 < t < 1 or k =,±1,±2,±3, ...

$$X(z) = Z[x(kT)] = \sum_{k=-\infty}^{\infty} x(kT)z^{-k}$$

The one sided z-transform has a convenient closed form solution in its region of convergence for most engineering applications. Whenever X(z), an infinite series in z^{-1}, converges outside the circle |z| = R, where R is the radius of absolute convergence it is not needed each time to specify the values of z over which X(z) is convergent.

I.e. for |z| > R ⟹convergent and for |z| < R ⟹ divergent.

Z-Transforms of some elementary functions:

(i) Unit step function is defined as:

$$u_s(t) = \begin{cases} 0, & for\ t < 0 \\ 1, & for\ t \geq 0 \end{cases}$$

Unit step sequence is defined as

$$u_s(kT) = \begin{cases} 0, & for\ k < 0 \\ 1, & for\ k = 0,1,2.. \end{cases}$$

Assuming that the function is continuous from right, the Z-transform is:

$$U_s(z) = \sum_{k=0}^{\infty} u_s(kT) z^{-k} = \sum_{k=0}^{\infty} z^{-k} = 1 + z^{-1} + z^{-2} + z^{-3} + \cdots = \frac{1}{1 - z^{-1}} = \frac{z}{z - 1}$$

The above series converges if $|z| > 1$.

(ii) Unit ramp function is defined as:

$$u_r(t) = \begin{cases} 0, & for\ t < 0 \\ t, & for\ t \geq 0 \end{cases}$$

Again $\quad u_r(kT) = kT, \qquad k = 0,1,2,\ldots$

The Z-transform is:

$$U_r(z) = \sum_{k=0}^{\infty} u_r(kT) z^{-k} = \sum_{k=0}^{\infty} kT z^{-k} = \frac{Tz}{(z-1)^2}$$

The above series converges if $|z| > 1$.

(iii) Exponential function is defined as:

$$x(t) = \begin{cases} 0, & for\ t < 0 \\ e^{-at}, & for\ t \geq 0 \end{cases}$$

Again $\quad x(kT) = e^{-akT}, \qquad k = 0,1,2,\ldots$

The Z-transform is:

$$X(z) = \sum_{k=0}^{\infty} x(kT) z^{-k} = \sum_{k=0}^{\infty} e^{-akT} z^{-k} = \frac{z}{(z - e^{-aT})}$$

Lecture Note 4(Theorems & properties of Z-Transform)

Table for z- and s-transform:

	$f(t)$	$F(s)$	$F(z)$
1.	$u(t)$	$\frac{1}{s}$	$\frac{z}{z-1}$
2.	t	$\frac{1}{s^2}$	$\frac{Tz}{(z-1)^2}$
3.	t^n	$\frac{n!}{s^{n+1}}$	$\lim_{a \to 0} (-1)^n \frac{d^n}{da^n}\left[\frac{z}{z-e^{-aT}}\right]$
4.	e^{-at}	$\frac{1}{s+a}$	$\frac{z}{z-e^{-aT}}$
5.	$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$	$(-1)^n \frac{d^n}{da^n}\left[\frac{z}{z-e^{-aT}}\right]$
6.	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$	$\frac{z \sin \omega T}{z^2 - 2z\cos \omega T + 1}$
7.	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$	$\frac{z(z-\cos \omega T)}{z^2 - 2z\cos \omega T + 1}$
8.	$e^{-at}\sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$	$\frac{ze^{-aT}\sin \omega T}{z^2 - 2ze^{-aT}\cos \omega T + e^{-2aT}}$
9.	$e^{-at}\cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$	$\frac{z^2 - ze^{-aT}\cos \omega T}{z^2 - 2ze^{-aT}\cos \omega T + e^{-2aT}}$

Important properties & theorems of z-transform:

1. Multiplication by a constant: $Z[ax(t)] = aX(z)$, where $X(z) = Z[x(t)]$

2. Linearity: If $x(k) = \alpha f(k) \pm \beta g(k)$, then $X(z) = \alpha F(z) \pm \beta G(z)$

3. Multiplication by a^k : $Z[a^k x(k)] = X(a^k z)$

4. Real shifting: $Z[x(t-nT)] = z^{-n}X(z)$ and

$$Z[x(t+nT)] = z^{n}\left[X(z) - \sum_{k=0}^{\infty} x(kT) z^{-k}\right]$$

5. Complex shifting: $Z[e^{\pm at}x(t)] = X(ze^{\mp at})$

6. Initial value theorem:

$$x(0) = \lim_{z\to\infty} X(z)$$

7. Final value theorem:

$$\lim_{k\to\infty} x(k) = \lim_{z\to 1}[(1 - z^{-1})X(z)]$$

Lecture Note 5(Inverse Z-Transform)

Inverse Z-transforms:

When F(z), the Z-transform of f(kT) or f(k), is given, the operation that determines the corresponding x(kT) or x(k) is called inverse Z-transform. It should be noted that only the time sequence at the sampling instants is obtained from the inverse Z-transform. So the inverse Z-transform of F(z) yield a unique f(k) but does not yield a unique f(t).

$$\Rightarrow f(kT) = Z^{-1}\left[F(z)\right]$$

The inverse Z- transform can be obtained by using

(i) Power series method:

Since F(z) is mostly expressed in the ratio of polynomial form

$$F(Z) = \frac{b_0 z^m + b_1 z^{m-1} + \cdots + b_m}{z^n + a_1 z^{n-1} + \cdots + a_n}$$

We can immediately recognize the sequence of f(k) if F(z) can be written in form of series with increasing powers of z^{-1} .This is easily done by dividing the numerator by the denominator.

I,e $F(z) = f(0) + f(1)z^{-1} + f(2)z^{-2} + \cdots + f(k)z^{-k} + \cdots$

Here the values of f(k) for k=0,1,2,… can be determined by inspection.

(ii) Partial fraction expansion method:

To expand F(z) into partial fractions, we first factor the denominator polynomial of F(z) and find the poles F(z):

$$F(z) = \frac{b_0 z^m + b_1 z^{m-1} + \cdots + b_m}{(z-p_1)(z-p_2)\dots(z-p_n)}$$

We then expand F(z)/z into partial fractions so that each term is easily recognizable in a table of Z-transforms. I.e.

$$\frac{F(z)}{z} = \frac{a_1}{z-p_1} + \frac{a_2}{z-p_2} + \cdots + \frac{a_n}{z-p_n}$$

Where $a_t = \left[(z - p_t)\frac{F(z)}{Z}\right]_{z=p_t}$

(iii) Inversion integral method:

The formula for inversion integral method is given by

$$Z^{-1}[F(z)] = f(k) = \frac{1}{2\pi j}\oint_C X(z)z^{k-1}dz$$

Where C is a circle with its center at the origin of the z plane such that all poles of $X(z)z^{k-1}$ are inside it.

Z-transform method for solving Difference Equation:

One of the most important applications of Z-transform is in the solution of linear difference equations. Let us consider that a discrete time system is described by the following difference equation.

$$x(k+2) + 3x(k+1) + 2x(k) = 0, \qquad x(0) = 0, \qquad x(1) = 1$$

Taking the Z-transform of both sides of the given difference equation, we obtain

$$[z^2X(z) - z^2X(0) - zX(1)] + 3[zX(z) - zX(0)] + 2X(z) = 0$$

Substituting the initial data and after simplifying we get

$$X(z) = \frac{z}{z^2 + 3z + 2} = \frac{z}{z+1} - \frac{z}{z+2}$$

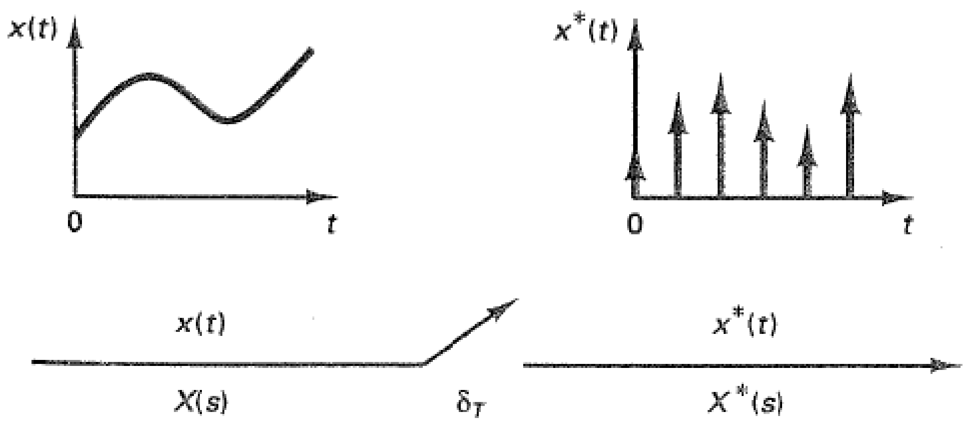
Taking the inverse Z-transform we get

$$x(k) = (-1)^k - (-2)^k, \qquad k = 0, 1, 2, \ldots$$

Lecture Note 6(Z-plane Analysis, sampling & hold)

Impulse Sampling:

In case of an impulse sampling, the output of the sampler is considered to be a train of impulses that begin with t=0, with the sampling period equal to T and the strength of each impulse equal to sampled value of the continuous-time signal at the corresponding sampling instant as shown in figure below.



(Impulse sampler)

The sampler output is equal to the product of the continuous-time input x(t) and the train of unit impulses $\delta_T(t)$.

The train of unit impulses $\delta_T(t)$ can be defined as

$$\delta_T(t) = \sum_{k=0}^{\infty} \delta(t - kT)$$

Now the sampler output can be expressed as

$$x^*(t) = \sum_{k=0}^{\infty} x(kT)\delta(t - kT)$$

$$\Rightarrow X^*(s) = \sum_{k=0}^{\infty} x(kT)e^{-kTs}$$

In the impulse sampler the switch may be thought of closing instantaneously every sampling period T and generating impulses x(kT)δ(t-kT).The impulse sampler is a fictitious sampler and it does not exist in the real world.

Data Hold:

Data-hold is a process of generating a continuous-time signal h(t) from a discrete-time sequence x(kT).A hold circuit converts the sampled signal into a continuous-time signal, which approximately reproduces the signal applied to the sampler.

The signal h(t) during the interval $kT \leq t \leq (k+1)T$ can be expressed as

$$h(kT + \tau) = a_n\tau^n + a_{n-1}\tau^{n-1} + \cdots + a_1\tau + a_0 \qquad where\ 0 \leq \tau \leq T$$

Since, $h(kT) = x(kT)$

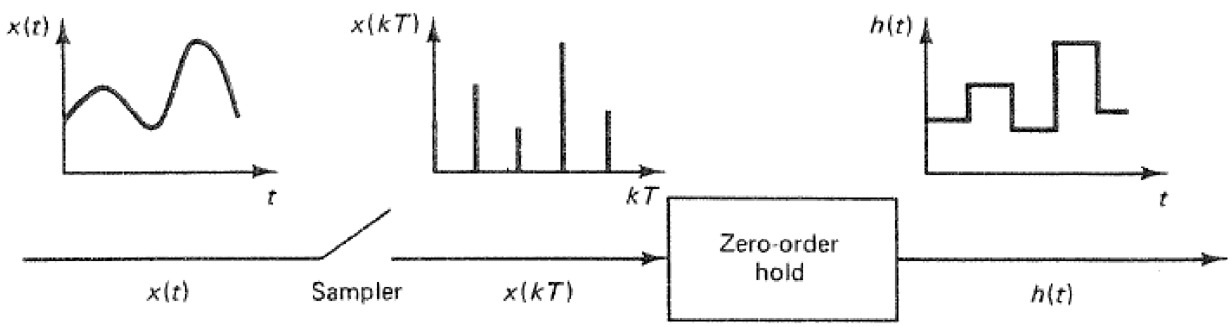
So,

$$h(kT + \tau) = a_n\tau^n + a_{n-1}\tau^{n-1} + \cdots + a_1\tau + x(kT) \qquad \rightarrow (1)$$

If the data-hold circuit is an nth-order polynomial extrapolator, it is called an nth order hold.

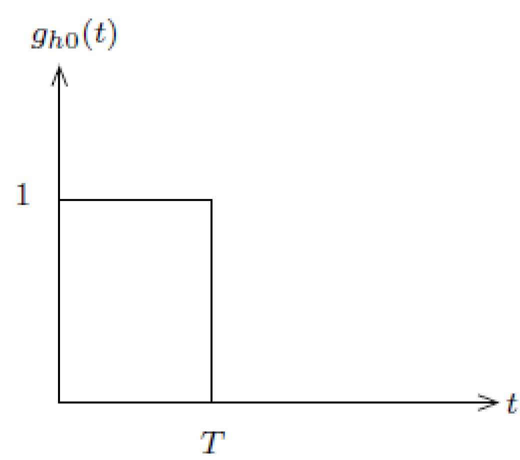
Zero-Order Hold:

It is the simplest data-hold obtained by putting n=0 in eq(1), gives $h(kT + \tau) = x(kT)$.It indicates that the circuit holds the value of h(kT) for $kT \leq t < (k + 1)T$ until the next sample h((k + 1)T) arrives.



(sampler and zero order hold)

The accuracy of zero order hold (ZOH) depends on the sampling frequency. When T→0, the output of ZOH approaches the continuous time signal. Zero order hold is again a linear device which satisfies the principle of superposition.



(Impulse response of ZOH)

The impulse response of a ZOH, as shown in the above figure can be written as

$$g_{ho}(t) = u_s(t) - u_s(t - T)$$

$$\Rightarrow G_{ho}(s) = \frac{1 - e^{-Ts}}{s}$$

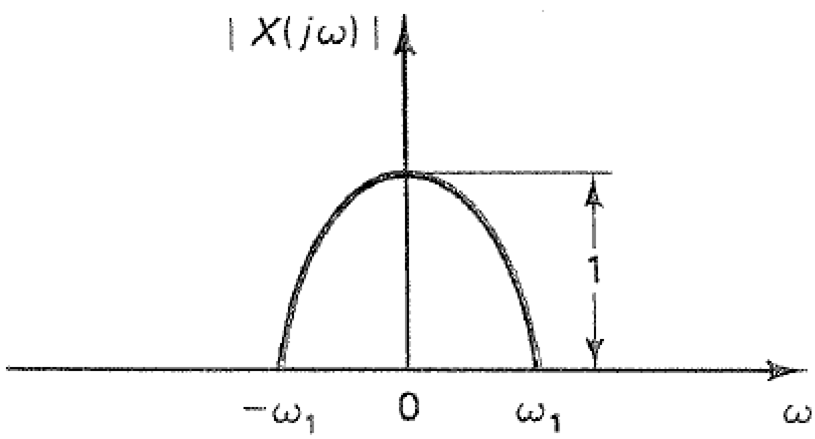
Lecture Note 7(Reconstruction of signal)

Data Reconstruction:

To reconstruct the original signal from a sampled signal, there is a certain minimum frequency that the sampling operation must satisfy. Such a minimum frequency is specified by the sampling theorem.

Sampling Theorem:

If the sampling frequency $\omega_s = 2\pi/T$ (where T is the sampling period) and ω_1 is the highest



frequency component present in the continuous- time signal x(t),then it is theoretically possible that the signal x(t) can be constructed completely from the sampled signal x*(t) if

$\omega_s > 2\,\omega_1$.

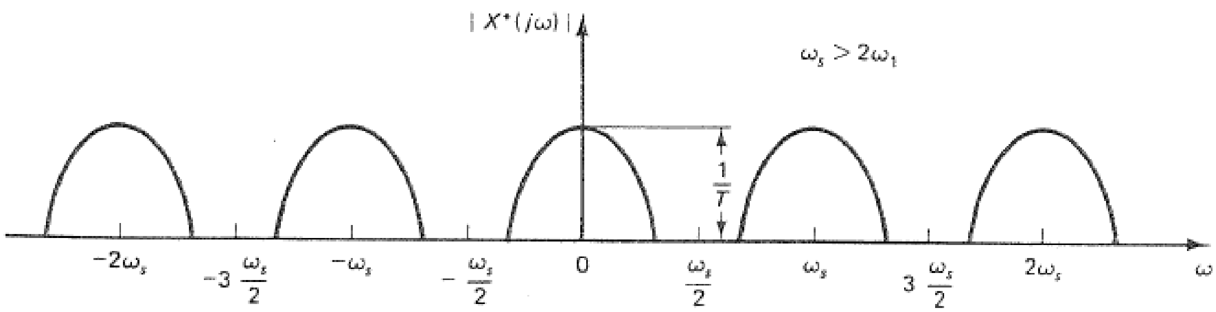
Considering the frequency spectrum of the signal x(t) as shown in the figure below

Then the frequency spectra of the sampled signal can be expressed as

$$X^*(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j\omega + j\omega_s k) = \cdots + \frac{1}{T}X(j(\omega - \omega_s)) + \frac{1}{T}X(j\omega) + \frac{1}{T}X(j(\omega + \omega_s)) + \cdots$$

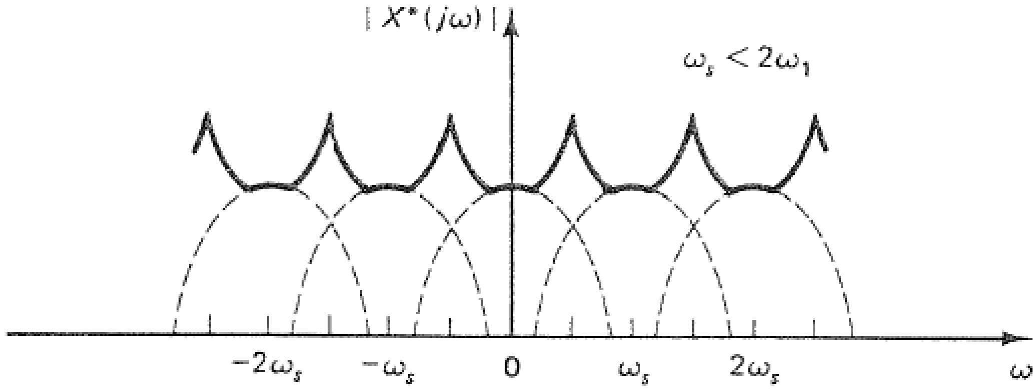
The above equation gives the frequency spectrum of the sampled signal x*(t) as shown in

figure below in which we can see that frequency spectrum of the impulse sampled signal is reproduced an infinite number of times and is attenuated by the factor 1/T.



The above figure represents the frequency spectrum for $\omega_s > 2\ \omega_1$ without any overlap and the continuous time signal x(t) can be reconstructed from the impulse-sampled signal $x^\star(t)$.

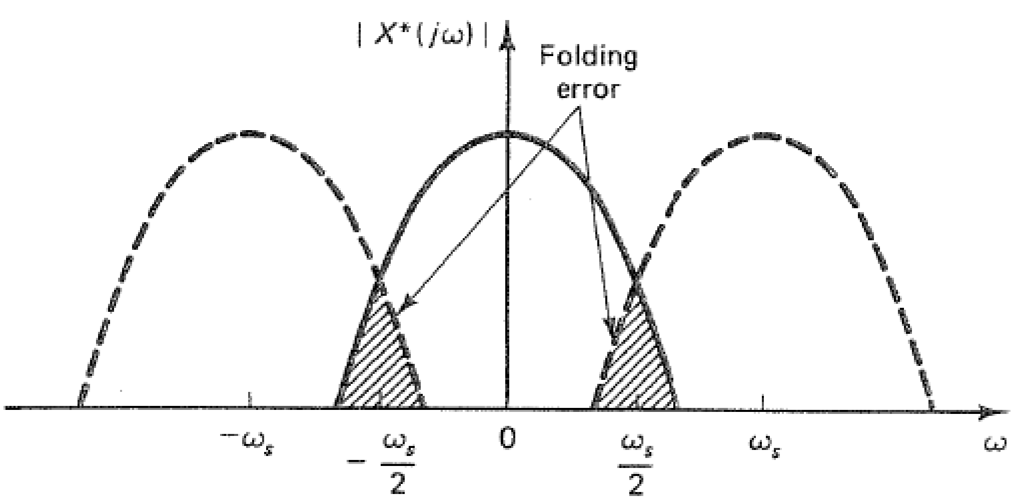
For $\omega_s < 2\ \omega_1$,as shown in figure below, the original shape of $|X(j\omega)|$ no longer appears in the plot of $|\ X^\star(j\omega)|$ because of the superposition of the spectra. As a result the continuous time signal x(t) can not be reconstructed from the impulse-sampled signal $x^\star(t)$.



Lecture Note 8(Folding & Aliasing)

Folding:

The overlapping of the high frequency components with the fundamental component in the frequency spectrum is sometimes referred to as folding and the frequency $\omega_s/2$ is often known as folding frequency. The frequency ω_s is called Nyquist frequency.. The figure



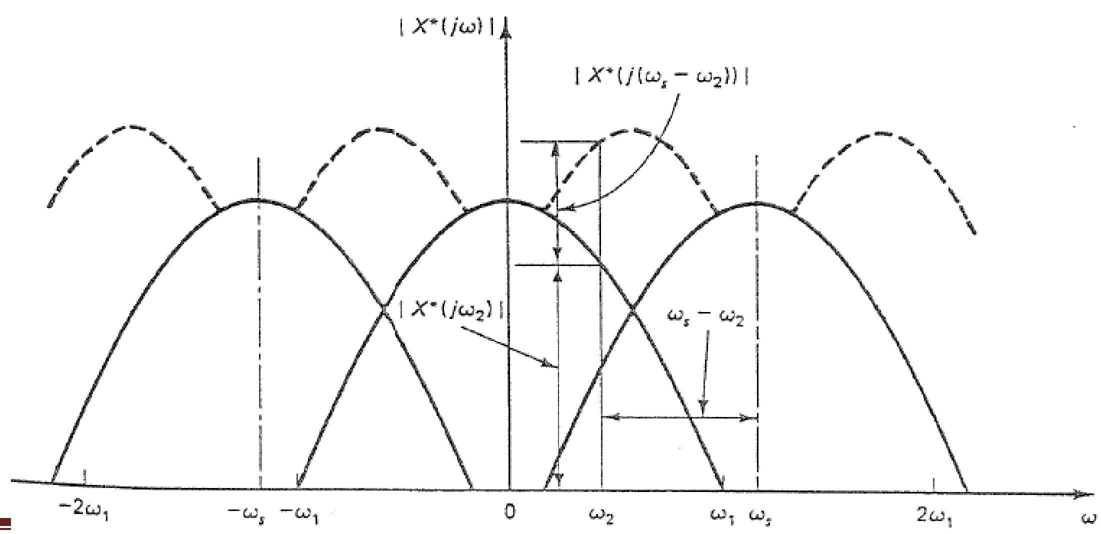
below shows the region where folding error occurs.

Aliasing:

If the sampling rate is less than twice the input frequency, the output frequency will be different from the input which is known as **aliasing**. The output frequency in that case is called **alias frequency** and the period is referred to as **alias period.**

Figure-1 shows the frequency spectra of an impulse-sampled signal $x^*(t)$, where $\omega_s < 2\,\omega_1$.

Now the frequency spectrum at an arbitrary frequency ω_2 in the overlap region includes components not only at frequency ω_2 but also at frequency $\omega_s - \omega_2$ (in general $n\omega_s \pm \omega_2$, where n is an integer).



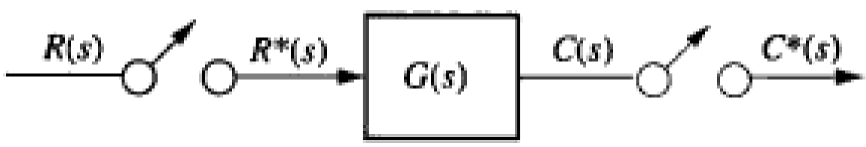
(Figure-1:frequency spectra of an impulse-sampled signal)

When the composite spectrum is filtered by a low pass filter, the frequency component at $\omega=n\omega_s \pm \omega_2$ will appear in the output as if it were a frequency component at $\omega=\omega_2$. The frequency $\omega=n\omega_s \pm \omega_2$ is called an alias of ω_2 and this phenomena is called aliasing.

Lecture Note 9(Pulse Transfer function)

Pulse Transfer Function:

The Pulse transfer function relates z-transform of the output at the sampling instants to the Z-transform of the sampled input.



(Block Diagram of a system with sampled input and output)

Now

$$c^*(t) = g^*(t) \quad (when\ r^*(t)\ is\ an\ impulse\ function)$$

$$\Rightarrow c^*(t) = \sum_{k=0}^{\infty} g(kT)\delta(t-kT)$$

Again

$$c(t) = r(0)g(t) + r(T)g(t-T) + \cdots$$

$$\Rightarrow c(kT) = r(0)g(kT) + r(T)g((k-1)T) + \cdots$$

$$\rightarrow c(kT) = \sum_{n=0}^{k} r(nT)g(kT-nT)$$

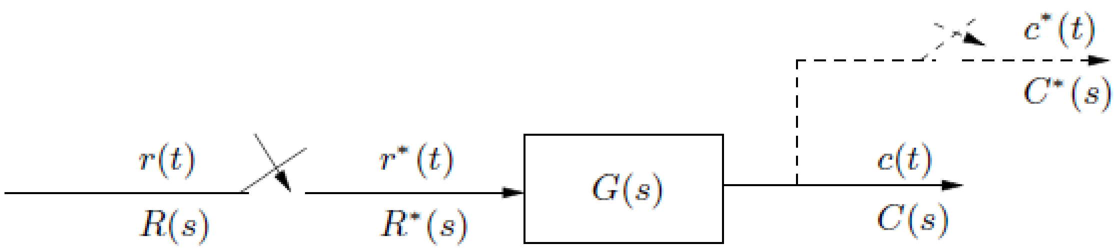
$$\Rightarrow C(z) = \sum_{k=-\infty}^{\infty} \sum_{n=0}^{k} r(nT)g(kT - nT)z^{-k}$$

Using real convolution theorem

$$C(z) = R(z)G(z) \Rightarrow G(z) = \frac{C(z)}{R(z)}$$

G(z) as given by the above equation ,the ratio of output C(z) and the input R(z) is called the *pulse transfer function*.

Starred Laplace Transform of the Signal involving Both Ordinary and starred Laplace Transform:



(Block Diagram of an Impulse sampled system)

The output of the system is $C(s) = G(s)R^{*}(s)$. The transfer function of the above system is difficult to manipulate because it contains a mixture of analog and digital components. Thus,it is desirable to express the system characteristics by a transfer function that relates $r^{*}(t)$ to $c^{*}(t)$, a fictitious sampler output as shown in the above figure.

Now

$$C^*(s) = \sum_{k=0}^{\infty} c(kT)e^{-kTs}$$

Since c(kT) is periodic,

$$C^*(s) = \frac{1}{T}\sum_{n=-\infty}^{\infty} C(s + nj\omega_s) \qquad with\ c(0) = 0$$

Siliarly,

$$R^*(s) = \frac{1}{T}\sum_{n=-\infty}^{\infty} R(s + nj\omega_s)$$

Again,

$$C^*(s) = \frac{1}{T}\sum_{n=-\infty}^{\infty} C(s + nj\omega_s) = \frac{1}{T}\sum_{n=-\infty}^{\infty} R^*(s + nj\omega_s)G(s + nj\omega_s)$$

Since $R^*(s)$ is periodic, $R^*(s + nj\omega_s) = R^*(s)$ .Thus

$$C^*(s) = \frac{1}{T}\sum_{n=-\infty}^{\infty} R^*(s)G(s + nj\omega_s) = \frac{R^*(s)1}{T}\sum_{n=-\infty}^{\infty} G(s + nj\omega_s)$$

By defining

$$G^*(s) = \frac{1}{T}\sum_{n=-\infty}^{\infty} G(s + nj\omega_s),$$

Then

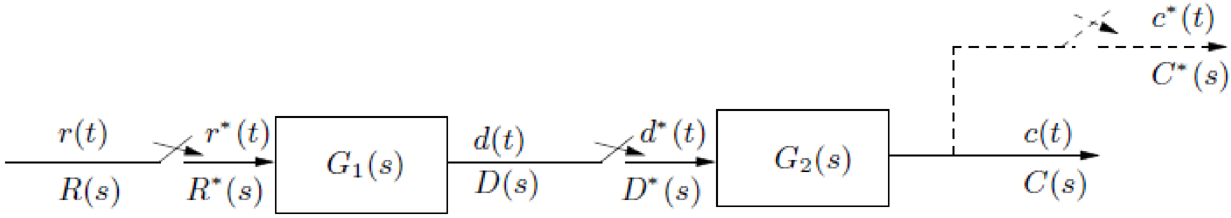
$$C^*(s) = R^*(s)G^*(s) \rightarrow G^*(s) = \frac{C^*(s)}{R^*(s)}$$

The above equation is referred as the starred transfer function. By substituting $z = e^{Ts}$ in the previous expression we will directly get the z-transfer function G(z) as

$$G(z) = \frac{C(z)}{R(z)}$$

Lecture Note 10(Pulse T.F. of Open Loop & Closed Loop Systems)

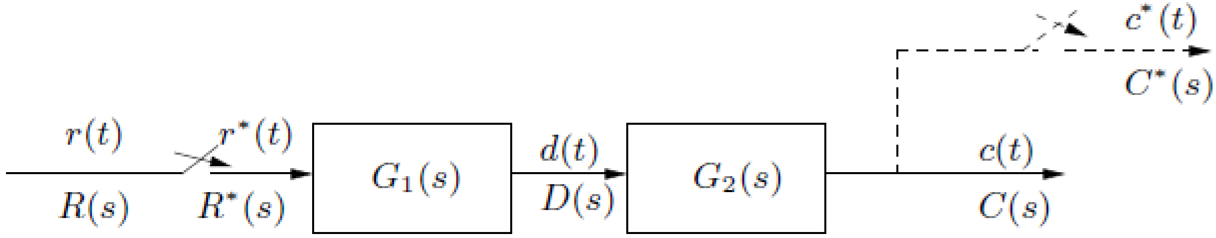
Pulse Transfer function of Cascaded Elements:



(Cascaded elements separated by a sampler)

The input-output relations of the two systems G_1 and G_2 are described by $D(z)=G_1(z)R(z)$ and $C(z) = G_2(z)D(z)$, So the input-output relation of the overall system is $C(z) = G_1(z)G_2(z)R(z)$.

So it can be concluded that the z-transfer function of two linear system separated by a sampler are the products of the individual z-transfer functions.



(Cascaded elements not separated by a sampler)

Here the input-output relations of the two systems G_1 and G_2 are described by

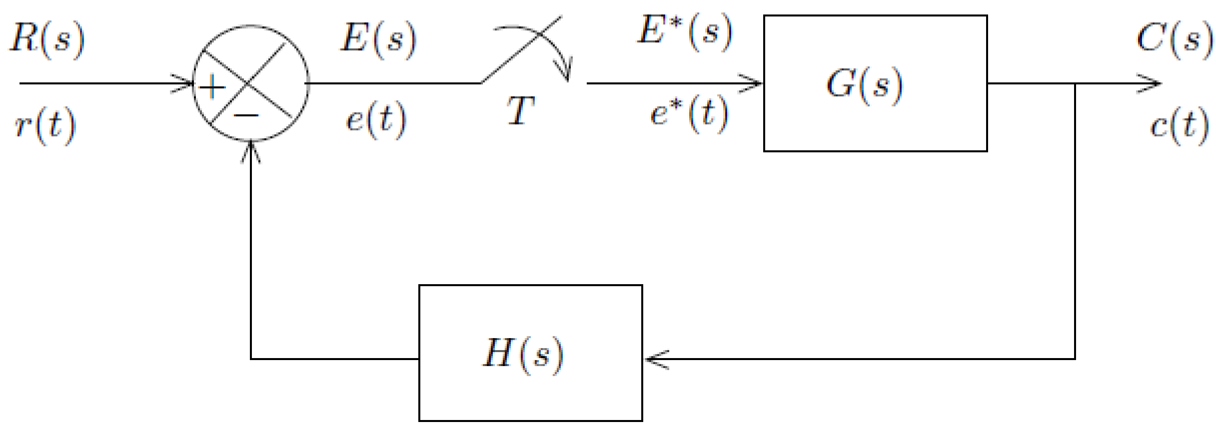
$C(s) = G_1(s)G_2(s)R^*(s)$ and the output of the fictitious sampler is

$C(z) = Z\,[G_1(s)G_2(s)]R(z)$

z-transform of the product $G_1(s)G_2(s)$ is denoted as $Z\,[G_1(s)G_2(s)] = G_1G_2(z) = G_2G_1(z)$

One should note that in general $G_1G_2(z) \neq G_1(z)G_2(z)$, except for some special cases. The overall output is thus, $C(z) = G_1G_2(z)R(z)$.

Pulse Transfer Functions of Closed Loop Systems:



(closed loop system with a sampler in the forward path)

For the above closed loop system, the output of the sampler is regarded as an input to the system. The input to the sampler is regarded as another output. Thus the input-output relations can be formulated as

$E(s) = R(s) - G(s)H(s)E^*(s)$

Taking starred Laplace transform on both sides

$E^*(s) = R^*(s) - GH^*(s)E^*(s)$

Or

$E^*(s) = \frac{R^*(s)}{1 + GH^*(s)}$

Since

$C^*(s) = G^*(s)E^*(s)$ so $C^*(s) = \frac{G^*(s)R^*(s)}{1 + GH^*(s)}$

$\Rightarrow \frac{C^*(s)}{R^*(s)} = \frac{G^*(s)}{1 + GH^*(s)}$ or $\frac{C(z)}{R(z)} = \frac{G(z)}{1 + GH(z)}$

Lecture Note 11(Mapping between s & z-plane)

Mapping between s-plane and z-plane:

The absolute stability and relative stability of the LTI continuous-time closed loop control system are determined by the locations of closed loop poles in the s-plane. Since the complex variables z and s are related by $= e^{Ts}$, the pole and zero locations in the z-plane are related to the pole and zero locations in the s-plane. So the stability of LTI discrete-time time closed loop control system are determined in terms of location of poles of the closed-loop pulse transfer function.

A pole in the s plane can be located in the z plane through the transformation $z = e^{Ts}$

Since $s = \sigma + j\omega$

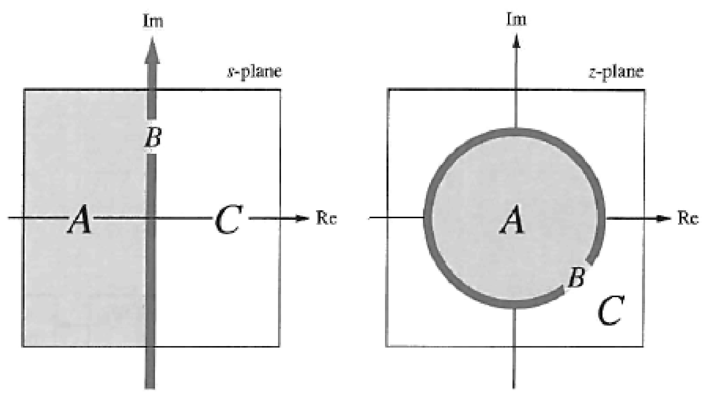
So $z = e^{T(\sigma + j\omega)} = e^{T\sigma} e^{jT\omega} = e^{T\sigma} (\cos \omega T + j \sin \omega T) = e^{T\sigma} \angle \omega T$

As σ is negative in the left half of the s-plane, so the left half of s-plane corresponds to

$|z| = e^{T\sigma} < 1$ (shown by region A inside of an unit circle).

For the $j\omega$ axis $\sigma = 0$,it corresponds to $|z| = 1$ .That is the imaginary axis in the s-plane corresponds to the unit circle in the z-plane (shown by region B).

For positive values of σ in the right half of the s-plane corresponds to



$|z| = e^{T\sigma} > 1$.(shown by region C outside the unit circle)

(mapping regions of s-plane on to the z-plane)

Thus, a digital control system is

(1) Stable if all poles of the closed-loop transfer function are inside the unit circle on the z-plane,

(2) Unstable if any pole is outside the unit circle and/or there are poles of multiplicity greater than one on the unit circle, and

(3) Marginally stable if poles of multiplicity one are on the unit circle and all other poles are inside the unit circle.

Lecture Note 12(Stability Analysis)

Stability Analysis of closed loop system in z-plane:

Consider the following closed loop pulse transfer function of an linear time invariant single-input-single-output discrete time control system

$$\frac{C(z)}{R(z)} = \frac{G(z)}{1+G(z)}$$

The stability of the above system can be determined from the location of closed loop poles in z-plane which are the roots of the characteristic equation

$$1 + GH(z) = 0$$

1. For the system to be stable, the closed loop poles or the roots of the characteristic equation must lie within the unit circle in z-plane. Otherwise the system would be unstable.

2. If a simple pole lies at $|z| = 1$, the system becomes marginally stable. Similarly if a pair of complex conjugate poles lies on the unit circle in the z-plane, the system is marginally stable. Multiple poles on unit circle make the system unstable.

Lecture Note 13(Routh Stability criterion)

Stability Analysis by use of Bilinear Transformation and Routh Stability Criterion:

It is a frequently used method in stability analysis of discrete time system by using bilinear transformation coupled with Routh stability criterion. This requires ransformation from z-plane to another plane called w-plane.

The bilinear transformation defined by

$$z = \frac{aw + b}{cw + d}$$

where a, b, c, d are real constants. If we consider a = b = c = 1 and d = −1, then the transformation takes a form

$$z = \frac{w + 1}{w - 1}$$

or,

$$w = \frac{z+1}{z-1}$$

This transformation maps the inside of the unit circle in the z-plane into the left half of the w-plane. Let the real part of w be α and imaginary part be β, so that $w = \alpha + j\beta$. The inside of the unit circle in z-plane can be represented by

$$|z| = \left|\frac{w+1}{w-1}\right| = \left|\frac{\alpha + j\beta + 1}{\alpha + j\beta - 1}\right| < 1$$

$$\Rightarrow \frac{(\alpha+1)^2 + \beta^2}{(\alpha+1)^2 - \beta^2} < 1 \Rightarrow (\alpha+1)^2 + \beta^2 < (\alpha+1)^2 - \beta^2 \Rightarrow \alpha < 0$$

Thus inside of the unit circle in z-plane maps into the left half of w-plane and outside of the unit circle in z-plane maps into the right half of w-plane. Although w-plane seems to be similar to s-plane, quantitatively it is not same.

In the stability analysis using bilinear transformation, we first substitute $z = \frac{w+1}{w-1}$ in the characteristics equation

$$P(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z^1 + a_n = 0$$

and simplify it to get the characteristic equation in w-plane

$$Q(w) = b_0 w^n + b_1 w^{n-1} + \cdots + b_{n-1} w^1 + b_n = 0$$

Once the characteristics equation P(z)=0 is transformed as Q(w) = 0, Routh stability Criterion is applied in the same manner as in continuous time systems.

Lecture Note 14(Jury Stability Test)

Jury Stability Test:

Assume that the characteristic equation P(z) is a polynomial in z as follows,

$$P(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z^1 + a_n$$

Where $a_0 > 0$ .Then the Jury table can be formed as shown below

Row	z^0	z^1	$\cdots$	z^{n-1}	z^n
1	a_n	a_{n-1}	$\cdots$	a_1	a_0
2	a_0	a_1	$\cdots$	a_{n-1}	a_n
3	b_{n-1}	b_{n-2}	$\cdots$	b_0	
4	b_0	b_1	$\cdots$	b_{n-1}	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
2n-5	p_3	p_2	$\cdots$		
2n-4	p_0	p_1	$\cdots$		
2n-3	q_2	q_1	q_0		

Where,

$$b_k = \begin{vmatrix} a_n & a_{n-1-k} \\ a_0 & a_{k+1} \end{vmatrix}, \qquad k = 0,1,2,\ldots,n-1$$

$$c_k = \begin{vmatrix} b_{n-1} & b_{n-2-k} \\ b_0 & b_{k+1} \end{vmatrix}, \qquad k = 0,1,2,\ldots,n-2$$

$$q_k = \begin{vmatrix} p_3 & p_{2-k} \\ p_0 & p_{k+1} \end{vmatrix}, \qquad k = 0,1,2$$

This system will be stable if the following conditions are all satisfied:

1. $|a_n| < a_0$
2. $[P_z(z)]_{z=1} > 0$
3. $[P_z(z)]_{z=1} > 0$ for n even and $[P_z(z)]_{z=-1} < 0$ for odd
4. $|b_{n-1}| < |b_0|$, $|c_{n-2}| < |c_0|$, ……., $|q_2| < |q_0|$

Module – 2: State Space Analysis

Lecturer: 1

: Introduction to State Space analysis

: Understanding the Concepts of State, State Variable and State Model of continuous time system

: State Model of Linear systems

Analysis and design of the non-linear, time variant, multivariable continuous system can be easily done using Modern control system i.e. state – space – based methods, where system model are directly written in the time domain. Its performance in real application where plant model is uncertain has been improved using robust control system, i.e. combining the modern state space and classical frequency domain techniques.

In this chapter, a basic study of time domain analysis of linear, time invariant, continuous system for single input single output (SISO) system has been studied, using state space based method.

Defining the following basic terms:

State: State represents the status of a dynamical system, with the minimum set of variables (known as state variables) such as the knowledge of these variables at $t=t_o$, together with the knowledge of the inputs for $t \geq t_o$, completely determines the behavior of the system for $t > t_o$.

State Variable: The variables that represent the status of the system at any time t, are called state variable.

State vector: A set of state variable expressed in a matrix is called state vector.

State space: Any n-dimensional state vector determines a point (called the state point) in an n-dimensional space called the state space.

State Trajectory: The curve traced out by the state point from $t = t_0$ to $t = t_1$ in the direction of increasing time is known as the state trajectory.

State Model: The state vectors with input/output equations constitute the state model of the system.

State Model of a Continuous time-invariant system:

$\dot{x}(t) = Ax(t) + Bu(t)$; State equation ……… (1)

$y(t) = Cx(t) + Du(t)$; Output equation ……… (2)

Where $x(t)$ is $n \times 1$ state vector, $u(t)$ is $m \times 1$ input vector, A is $n \times n$ system matrix and B is $n \times m$ input matrix. Similarly $y(t)$ is $p \times 1$ output vector, C is $p \times n$ output matrix and D is $p \times m$ transmission matrix.

Lecturer: 2

: State Model of Linear SISO systems

: Linearization of the State Equation

Linearization of the state Equation:

The state equation $\dot{x} = f(x, u)$ of a general time – invariant, non-linear continuous system can be linearized, by considering small variations about an equilibrium point (x_o, u_o) i.e. $\dot{x} = f(x, u) = 0$, where x_o , u_o is assumed initial state point and input vector respectively. is state space.

Since the derivatives of all the state variables are zero at the equilibrium point, the system continues to lie at the equilibrium point unless otherwise disturbed. The state equation can be linearized about the operating point (x_o, u_o) by expanding it into Talyor series and neglecting terms of second and higher order.

The linearized component equation can be written as the vector matrix equation

$\dot{\tilde{x}} = A\tilde{x} + B\tilde{u}$

Where

A=

$\cdots \& \cdots @ \frac{\partial f_n}{\partial x_2} \& \frac{\partial f_n}{\partial x_n}$

, B =

$\cdots \& \cdots @ \frac{\partial f_n}{\partial u_2} \& \frac{\partial f_n}{\partial u_n}$

All the partial derivatives in the matrices A and B defined above (called the Jacobian matrices) are evaluated at the equilibrium state (x_o, u_o).

Lecturer: 3

: State space representation using physical variable

Introduction: In this representation the state variables are real physical variables, which can be measured and used for manipulation or for control purposes. The approach generally adopted is to break the block diagram of the transfer function into subsystems in such a way that the physical variables can be identified.

How to solve a problem

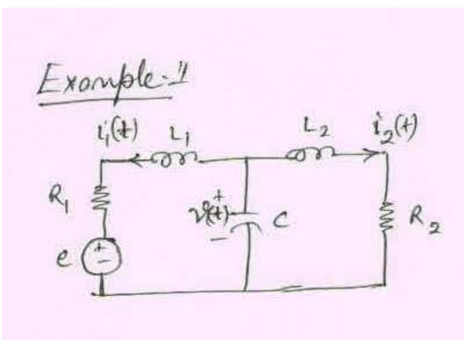
1. For an electrical network, apply any of the Network analysis (such as KCL, KVL, Mesh and Node).

2. Write the equations such that all the elements of the network have been considered.
3. The equation will be having derivative terms for energy storage elements.
4. Try to convert each equation to single order.
5. Finally, write the state space equation.

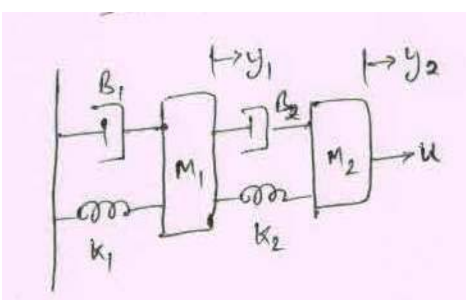
For a mechanical system

1. For Mechanical system, convert the physical Model into its equivalent electrical network using Force voltage or Force current analogy.
2. Electrical network can be solved as above.

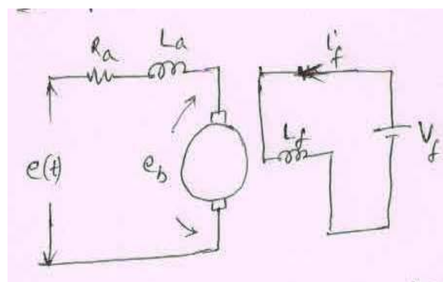
Example:1



Example: 2



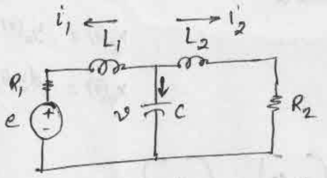
Example: 3



Solution: Example.1

State Space Representation Using Physical System :-

Ex 1



$x_1 = v(t)$

$x_2 = i_1(t)$

$x_3 = i_2(t)$

Eqn

$$\frac{dv}{dt} = \frac{-1}{C} i_1 - \frac{1}{C} i_2$$

$$\frac{di_1}{dt} = \frac{-R_1}{L_1} i_1 - \frac{e}{L_1} + \frac{v}{L_1}$$

$$\frac{di_2}{dt} = \frac{-R_2}{L_2} i_2 + \frac{v}{L_2}$$

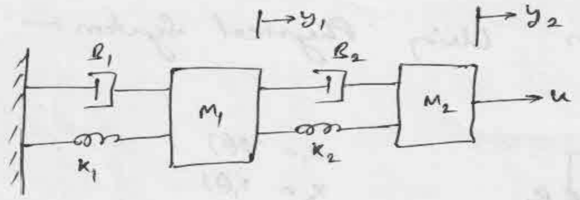
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & -1/C & -1/C \\ 1/L_1 & -R_1/L_1 & 0 \\ 1/L_2 & 0 & -\frac{R_2}{L_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ -1/L_1 \\ 0 \end{bmatrix} e$$

Assume the voltage across R_2 and current through R_2 are the output variable y_1 and y_2 respectively.

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & R_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Solution: Example.2,3

Ex 2



$x_1(t) = y_1(t)$

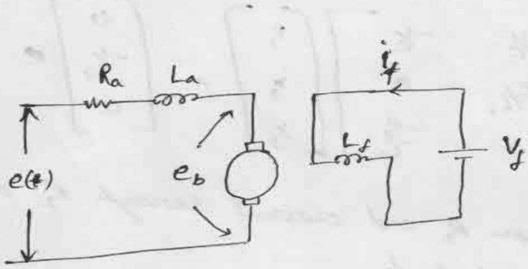
$x_2(t) = dy_1(t)/dt$

$x_3(t) = y_2(t)$

$x_4(t) = dy_2(t)/dt$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{(K_1+K_2)}{M_1} & -\frac{(B_1+B_2)}{M_1} & \frac{K_2}{M_1} & \frac{B_2}{M_1} \\ 0 & 0 & 0 & 1 \\ \frac{K_2}{M_2} & \frac{B_2}{M_2} & -\frac{K_2}{M_2} & -\frac{B_2}{M_2} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/M_2 \end{bmatrix} u$$

Ex 3



$x_1(t) = \theta(t)$

$x_2(t) = \dot{\theta}(t)$

$x_3(t) = i_a$

$$e_b = K_b \frac{d\theta}{dt}$$

$$J \frac{d^2\theta}{dt^2} + B \frac{d\theta}{dt} = K_T i_a$$

$$e(t) - e_b = R_a i_a + L_a \frac{di_a}{dt}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -B/J & K_T/J \\ 0 & -K_b/L & -\frac{R_a}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1/L \end{bmatrix} e(t)$$

Lecturer: 4

: State space representation using Phase Variable or Observable canonical form and Alternate Phase Variable or Controllable Canonical form.

: Phase variable formulations for transfer function with poles and zeros.

Introduction: It is often convenient to consider the output of the system as one of the state variable and remaining state variable as derivatives of this state variables. The state variables thus obtained from one of the system variables and its (n-1) derivatives, are known as n-dimensional phase variables.

Steps to be followed:

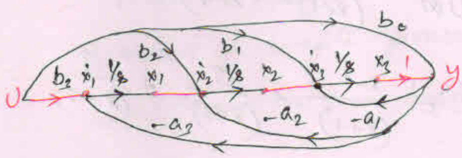
1. Obtain the differential equation for the given Physical Model.

2. Obtain the transfer function from the given differential equation.

An example has been solved both in phase and alternate phase variable form.

Alternate Phase variable Observable Canonical form

$$T(s) = \frac{b_0 + b_1/s + b_2/s^2 + b_3/s^3}{1 - (-a_1/s - a_2/s^2 - a_3/s^3)}$$



$$y = b_0 u + x_3$$

$$\dot{x}_3 = x_2 - a_1 y + b_1 u$$

$$\dot{x}_3 = -a_1 x_3 + x_2 + (b_1 - a_1 b_0) u$$

$$\dot{x}_2 = -a_2 x_3 + x_1 + (b_2 - a_2 b_0) u$$

$$\dot{x}_1 = -a_3 x_3 + (b_3 - a_3 b_0) u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} b_3 - a_3 b_0 \\ b_2 - a_2 b_0 \\ b_1 - a_1 b_0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [b_0] u$$

To determine the initial state values.

$$y = b_0 u + x_3$$

$$\Rightarrow \boxed{x_3 = y - b_0 u} \quad \text{or} \quad \boxed{\dot{x}_3 = \dot{y} - b_0 \dot{u}}$$

$$x_3(0) = y(0) - b_0 u(0)$$

Also, $\dot{x}_3 = x_2 - a_1 y + b_1 u$

$$\Rightarrow x_2 = \dot{x}_3 + a_1 y - b_1 u$$

$$= (\dot{y} - b_0 \dot{u}) + a_1 y - b_1 u$$

$$\boxed{x_2 = (a_1 y + \dot{y}) + (-b_1 u - b_0 \dot{u})}$$

Also. $\dot{x}_2 = x_1 - a_2 y + b_2 u$

$$x_1 = \dot{x}_2 + a_2 y - b_2 u$$

$$x_1 = (a_2 y + a_1 \dot{y} + \ddot{y}) + (-b_2 u - b_1 \dot{u} - b_0 \ddot{u})$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} a_2 & a_1 & 1 \\ a_1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} y(0) \\ \dot{y}(0) \\ \ddot{y}(0) \end{bmatrix} + \begin{bmatrix} -b_2 & -b_1 & b_0 \\ -b_1 & -b_0 & 0 \\ -b_0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u(0) \\ \dot{u}(0) \\ \ddot{u}(0) \end{bmatrix}$$

$$T(s) = \frac{b_0 s^3 + b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3}$$

Alternate Phase variable Controllable Canonical form.

$$T(s) = \left(\frac{1}{s^3 + a_1 s^2 + a_2 s + a_3} \right) \times \left(b_0 s^3 + b_1 s^2 + b_2 s + b_3 \right)$$

$$\frac{Y(s)}{U(s)} = \frac{X(s)}{U(s)} \cdot \frac{Y(s)}{X(s)}$$

$$\frac{X(s)}{U(s)} = \frac{1}{s^3 + a_1 s^2 + a_2 s + a_3}$$

or $s^3 X(s) + a_1 s^2 X(s) + a_2 s X(s) + a_3 X(s) = U(s)$

Let $X(s) = x_1(s)$, $sX(s) = x_2(s)$, $s^2 X(s) = x_3(s)$

$$\Rightarrow \dot{x}_3(s) = -a_1 x_3(s) - a_2 x_2(s) - a_3 x_1(s) + U(s)$$

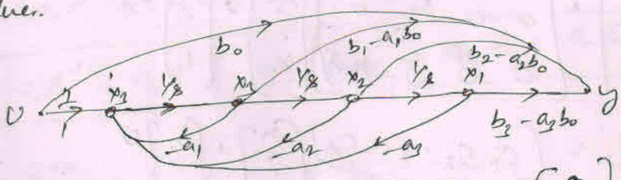
And

$$\frac{Y(s)}{X(s)} = b_0 s^3 + b_1 s^2 + b_2 s + b_3$$

or $Y(s) = b_0 \dot{x}_3(s) + b_1 x_3(s) + b_2 x_2(s) + b_3 x_1(s)$

or $Y(s) = b_0 \left(-a_1 x_3(s) - a_2 x_2(s) - a_3 x_1(s) + U(s)\right) + (\text{iii})$

or $Y(s) = (b_1 - a_1 b_0) x_3(s) + (b_2 - a_2 b_0) x_2(s) + b_0 U(s) + (b_3 - a_3 b_0) x_1(s)$



$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$Y = \begin{bmatrix} b_3 - a_3 b_0 & b_2 - a_2 b_0 & b_1 - a_1 b_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [b_0] u$$

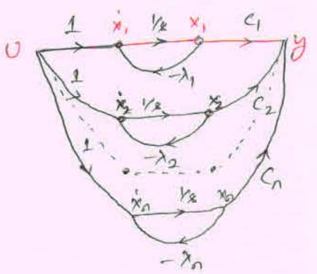
Lecturer: 5

: State space representation using Canonical Variables (Normal form or Jordan Normal Form Representation).

$$T(s) = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$$

$$\frac{Y(s)}{U(s)} = T(s) = b_0 + \sum_{i=1}^{n} \frac{c_i}{s-\lambda_i}$$

In case of Jordan block representation. (The denominator will have multiple roots.



$$\dot{x}_1 = u - \lambda_1 x_1$$
$$\dot{x}_2 = u - \lambda_2 x_2$$
$$\vdots$$
$$\dot{x}_n = u - \lambda_n x_n$$

$$y = c_1 \dot{x}_1 + c_2 \dot{x}_2 + \cdots + c_n x_n + b_0 u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \lambda_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0 u$$

Ex

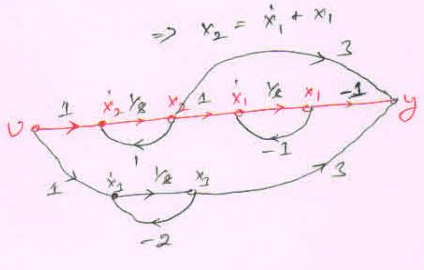
$$\frac{Y(s)}{U(s)} = \frac{2s^2 + 6s + 7}{(s+1)^2 (s+2)}$$

$$\frac{Y(s)}{U(s)} = \frac{-1}{s+1} + \frac{3}{(s+1)^2} + \frac{3}{s+2}$$

$$\Rightarrow \frac{X_2(s)}{U(s)} = \frac{1}{s+1}$$
$$\Rightarrow u = \dot{x}_2 + x_2$$

$$\Rightarrow \frac{X_3(s)}{U(s)} = \frac{1}{s+2}$$
$$\Rightarrow u = \dot{x}_3 + 2x_3$$

$$\Rightarrow \frac{X_1(s)}{U(s)} = \frac{X_2(s)}{s+1}$$
$$\Rightarrow x_2 = \dot{x}_1 + x_1$$



$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u$$

: Derivation of Transfer function for State Model

Consider the State Model

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

Taking Laplace transform on both sides yields,

$$sX(s) - X(0) = AX(s) + BU(s) \quad \text{--- (1)}$$

and

$$Y(s) = CX(s) + DU(s) \quad \text{--- (2)}$$

From eqn (1), $(sI - A)X(s) = X(0)\, BU(s)$

By applying matrix algebra on the Matrices A, B, C and D, we have,

$$X(s) = (sI-A)^{-1}X(0) + (sI-A)^{-1}BU(s) \quad \text{--- (3)}$$

Substituting the value of X(s) in eqn (2), we have.

$$Y(s) = C(sI-A)^{-1}X(0) + C(sI-A)^{-1}BU(s) + DU(s)$$

Assuming initial conditions to zero, we get,

$$Y(s) = C(sI-A)^{-1}BU(s) + DU(s)$$

$$T(s) = \frac{Y(s)}{U(s)} = C(sI-A)^{-1}B + D$$

$$= C\,\frac{\operatorname{adj}(sI-A)}{\det(sI-A)}\,B + D.$$

Lecturer: 6

: Eigen values and Eigenvectors

Introduction : - Eigen values and eigen vector are special form of square matrix. While the eigen value parameterizes the diagonal properties of the system (time scale, resonance properties, amplifying factor etc), the eigenvector define the vector coordinate of the normal mode of the system. Each eigenvector is associated with particular eigen values. The general state of the system can be expressed as a combination of eigenvector.

Eigen vector can be explained as a vector 'X ' such that matrix operator 'A' transforms it to a vector 'λX ' (λ is a constant) i.e. to a vector having the same direction in state space as the vector X .

Such a vector X is a solution of the equation

$AX = \lambda X$

Or $\lambda X \text{ I } AX = 0$ or $(\lambda I - A)X = 0$

Now, the above equation has a non-trivial solution of and only if

$(\lambda I - A) = 0$ or $q(\lambda) = \lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \ldots\ldots + a_n = 0$

Examples has been solved for different cases :

Ex 1

$\dot{x} = Ax + Bu$

$y = Cx + Du$

Let $A = \begin{bmatrix} 6 & 16 \\ -1 & -4 \end{bmatrix}$

The eigen values of an $n \times n$ matrix A are the roots of the characteristic equation

$$|\lambda I - A| = 0 \Rightarrow \begin{vmatrix} \lambda - 6 & -16 \\ 1 & \lambda + 4 \end{vmatrix} = 0 \Rightarrow \lambda = 4, -2$$

The eigen vector x_1 associated with eigenvalue $\lambda_1 = 4$ is

$$[\lambda_1 I - A][x_1] = 0$$

$$\begin{bmatrix} 4-6 & -16 \\ 1 & 4+4 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix} = 0 \Rightarrow -2x_{11} - 16x_{21} = 0 \; \& \; x_{11} - 8x_{21} = 0$$

If $x_{21} = 1$, $x_{11} = -8$ $\Rightarrow [x_1] = \begin{bmatrix} -8 \\ 1 \end{bmatrix}$

The eigenvector x_2 associated with eigenvalue $\lambda_2 = -2$ is

$$[\lambda_2 I - A][x_2] = 0$$

$$\begin{bmatrix} -2-6 & -16 \\ 1 & -2+4 \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{22} \end{bmatrix} = 0 \Rightarrow \begin{matrix} -8x_{12} - 16x_{22} = 0 \\ x_{12} + 2x_{22} = 0 \end{matrix}$$

If $x_{22} = 1$, $x_{12} = -2$ $\Rightarrow [x_2] = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$\therefore$ P matrix becomes $= \begin{bmatrix} -8 & -2 \\ 1 & 1 \end{bmatrix}$ $P^{-1} = \begin{bmatrix} 1 & 2 \\ -1 & 8 \end{bmatrix}$

And $P^{-1}AP$

$$= \begin{bmatrix} 8 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 6 & 16 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 8 \end{bmatrix}$$

(c) $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$ (Vander Monde Matrix)

$|\lambda I - A| = 0 \Rightarrow \lambda = -1, -2, -3$

$$P = \begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix}$$

$\therefore \hat{A} = P^{-1}AP = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$ $\hat{B} = P^{-1}B = \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix}$ $\hat{C} = CP = [1 \ 1 \ 1]$

Case 2 If the matrix A involves multiple eigenvalues, then diagonalization is impossible.

Then $P = \begin{bmatrix} 1 & 0 & 1 \\ \lambda_1 & 1 & \lambda_3 \\ \lambda_1^2 & 2\lambda_1 & \lambda_3^2 \end{bmatrix}$

And $P^{-1}AP = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

This is in the Jordan Canonical form.

Case 3:- If the roots are complex number.

$$\frac{1}{s^2 + a_2 s + a_3}$$

$$A = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -a_3 & -a_2 \end{bmatrix}$$

... the 'A' transform

Ex

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 3 & 0 & 2 \\ -12 & -7 & -6 \end{bmatrix} \qquad |\lambda I - A| = 0$$

$$\lambda_1 = -1,\ \lambda_2 = -2,\ \lambda_3 = -3$$

$$(\lambda_1 I - A) = \begin{bmatrix} -1 & -1 & 0 \\ -3 & -1 & -2 \\ 12 & 7 & 5 \end{bmatrix}$$

$$m_1 = \begin{bmatrix} C_{11} \\ C_{12} \\ C_{13} \end{bmatrix} = \begin{bmatrix} 9 \\ -9 \\ -9 \end{bmatrix} = \begin{bmatrix} 9 \\ -9 \\ -9 \end{bmatrix} \text{ or } m_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

The eigenvector m_1 could also be obtained from a solution of the homogeneous equations corresponding to the matrix $(\lambda I - A)x = 0$.

$$-x_1 - x_2 = 0$$
$$-3x_1 - x_2 - 2x_3 = 0$$
$$12x_1 + 7x_2 + 5x_3 = 0$$

Let $x_1 = 1$, $x_2 = -1$, $x_3 = \frac{3x_1 + x_2}{-2} = -1$

Solution of homogeneous equations is computationally more efficient than obtaining co-factors of row when the dimension of A is large.

Similarly,
$$(\lambda_2 I - A) = \begin{bmatrix} -2 & -1 & 0 \\ -3 & -2 & -2 \\ 12 & 7 & 4 \end{bmatrix} \qquad m_2 = \begin{bmatrix} 6 \\ -12 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix}$$

And
$$(\lambda_3 I - A) = \begin{bmatrix} -3 & -1 & 0 \\ -3 & -3 & -2 \\ 12 & 7 & 3 \end{bmatrix} \qquad m_3 = \begin{bmatrix} 5 \\ -15 \\ 15 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix}$$

$$M = \begin{bmatrix} 1 & 2 & 1 \\ -1 & -4 & -3 \\ -1 & 1 & 3 \end{bmatrix}$$

Generalized Eigenvectors

Case 2 → Eigenvalues having repeated roots.

$$A = \begin{bmatrix} 4 & 1 & -2 \\ 1 & 0 & 2 \\ 1 & -1 & 3 \end{bmatrix} \qquad \lambda_1 = 1, \quad \lambda_2 = 3, \quad \lambda_3 = 3$$

$$(\lambda_1 I - A) = \begin{bmatrix} -3 & -1 & 2 \\ -1 & 1 & -2 \\ -1 & 1 & -2 \end{bmatrix} \qquad m_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad m_1 = \begin{bmatrix} 0 \\ +8 \\ 4 \end{bmatrix}$$

$$(\lambda_2 I - A) = \begin{bmatrix} \lambda_2 - 4 & -1 & 2 \\ -1 & \lambda_2 & -2 \\ -1 & 1 & \lambda_2 - 3 \end{bmatrix} \qquad m_2 = \begin{bmatrix} \lambda_2^2 - 3\lambda_2 + 2 \\ \lambda_2 - 1 \\ \lambda_2 - 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} =$$

$$m_3 = \begin{bmatrix} \frac{d}{d\lambda_2} C_{11} \\ \frac{d}{d\lambda_2} C_{12} \\ \frac{d}{d\lambda_2} C_{13} \end{bmatrix} = \begin{bmatrix} 2\lambda_2 - 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

$$M = \begin{bmatrix} 0 & 2 & 3 \\ 8 & 2 & 1 \\ 4 & 2 & 1 \end{bmatrix}$$

: Time domain solution of State Equation

: Properties of the State Transition Matrix

Time domain Solution of State equation

In this section we will find the solutions of the state equations.

The state equation of a system is given by

$$\dot{X}(t) = AX(t) + Bu(t)$$

Before finding the solution, first let us assume a homogeneous state equation of the form,

$\frac{dx}{dt} = Ax$, $\frac{dx}{x} = A\,dt$, $\ln x = At$, $x = e^{At}$

$$\dot{x}(t) = AX(t) \quad \text{——— (1)}$$

Assume the solution is of the form,

$x(t) = e^{At}$

$$X(t) = b_0 + b_1 t + b_2 t^2 + \dots + b_k t^k + b_{k+1} t^{k+1} \dots \quad \text{——— (2)}$$

then we obtain,

$$\dot{X}(t) = b_1 + 2b_2 t + \dots + k b_k t^{k-1} + (k-1) b_{k+1} t^k + \dots \quad \text{——— (3)}$$

Substituting (2) and (3) in eq^n (1),

$$b_1 + 2b_2 t + \dots + k b_k t^{k-1} + (k-1) b_{k+1} t^k + \dots$$
$$= Ab_0 + Ab_1 t + Ab_2 t^2 + \dots + Ab_k t^k + Ab_{k+1} t^{k+1} \dots$$

Comparing the coefficients on both the sides,

$$b_1 = Ab_0$$

$$b_2 = \frac{1}{2}Ab_1 = \frac{1}{2}A^2b_0$$

$$b_3 = \frac{Ab_2}{3} = \frac{A}{3}\left(\frac{1}{2}A^2b_0\right) = \frac{A^3}{3!}b_0$$

$$\vdots$$

$$b_k = \frac{A^k}{k!}b_0$$

$$b_{k+1} = \frac{A^{k+1}}{(k+1)!}b_0$$

Substituting the above value in eq^n ②, we get,

$$X(t) = b_0 + b_1t + b_2t^2 + \dots + b_kt^k + b_{k+1}t^{k+1}$$

$$= b_0 + Ab_0t + \frac{1}{2}A^2b_0t^2 + \dots + \frac{A^k}{k!}b_0t^k + \frac{A^{k+1}}{(k+1)!}b_0t^{k+1}$$

$$= b_0\left(1 + At + \frac{A^2}{2!}t^2 + \frac{A^3t^3}{3!} + \dots + \frac{A^k}{k!}t^k + \dots \frac{A^{k+1}}{(k+1)!}t^{k+1}\right)$$

$$= b_0e^{At}$$

Substituting $t=0$, we get,

$$X(0) = b_0$$

$$\boxed{X(t) = e^{At}X(0)}$$

where e^{At} is a matrix known as state transition matrix and is denoted by $\phi(t)$ i.e,

$$\phi(t) = e^{At}.$$

Now Consider,

$$\dot{X}(t) = AX(t) + Bu(t)$$

or $$\dot{X}(t) - AX(t) = Bu(t)$$

Multiplying both sides with e^{-At}, we have,

$$e^{-At}\left[\dot{X}(t) - AX(t)\right] = e^{-At}Bu(t)$$

$$e^{-At}\dot{X}(t) - Ae^{-At}X(t) = e^{-At}Bu(t)$$

$$\frac{d}{dt}\left(e^{-At}X(t)\right) = e^{-At}Bu(t)$$

Integrating on both sides from 0 to t, we have,

$$e^{-At}X(t)\Big|_0^t = \int_0^t e^{-A\tau}Bu(\tau)\,d\tau$$

$$e^{-At}X(t) - X(0) = \int_0^t e^{-A\tau}Bu(\tau)\,d\tau$$

$$X(t) = e^{At}X(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau$$

$$= \underbrace{\phi(t)X(0)}_{\text{zero input component}} + \underbrace{\int_0^t \phi(t-\tau)Bu(\tau)\,d\tau}_{\text{zero state component}} \quad \text{--- (1)}$$

This is the desired solution.

The integral above is called as the convolution integral. The first term on the right hand side of the eq^n is the response due to the input $u(t)=0$. Hence it is the zero input response. The second term depends on the input only, not on the initial state vector. Hence it is known as zero state component.

If the initial state is known at $t=t_0$ rather than at

Then the above eq^n can be written as,

$$X(t) = e^{A(t-t_0)}X(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$$

⟹ Properties of the transition matrix

① $\phi(0)$ is an identity matrix :

$$x(t) = e^{At}x(0)$$
$$= \phi(t)x(0)$$

At $t=0$,
$$X(0) = \phi(0)X(0)$$

from which we have $\phi(0) = I$.

② $\phi(t) = e^{At} = (e^{-At})^{-1} = [\phi(-t)]^{-1}$

i.e $\phi(t) = [\phi(-t)]^{-1}$

Taking inverse on both side, we have

$$\phi^{-1}(t) = \phi(-t)$$

③ $\phi(t_1+t_2) = e^{A(t_1+t_2)} = \phi(t_1)\cdot\phi(t_2)$

④ $[\phi(t)]^n = (e^{At})^n = \phi(nt)$

i.e $[\phi(t)]^n = \phi(nt)$

10. **Laplace tranform Solution of State equation**

Consider the state equation; $\rightarrow$ $\dot{X} = AX + Bu$.

and the output equation,

$$Y = CX + DU.$$

Taking Laplace transform on both side yields,

$$sX(s) - X(0) = AX(s) + BU(s)$$

$$(sI - A)X(s) = X(0) + BU(s)$$

where I is the identity matrix.

$$X(s) = (sI - A)^{-1}[X(0) + BU(s)]$$

$$= \phi(s)[X(0) + BU(s)]$$

where $\phi(s) = (sI - A)^{-1}$

$$X(s) = \phi(s)X(0) + \phi(s)BU(s)$$

and $x(t) = \mathcal{L}^{-1}[\phi(s)X(0)] + \mathcal{L}^{-1}[\phi(s)BU(s)]$

(first term: zero-input component; second term: zero-state component.)

Lecturer: 8

: Computation of State transition matrix

(Time Domain Solution of State equation)

Q1 Determine the time response of a system given below:

$$\dot{x} = Ax \quad , \text{ where } A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \quad x(0) = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$$

and $y = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Solution

$$X(t) = \mathcal{L}^{-1}\{\phi(s)X(0)\} + \mathcal{L}^{-1}\{\phi(s)B\,U(s)\}$$

$$\phi(s) = (sI - A)^{-1} \quad , \quad \phi(t) = e^{At} \quad , \quad \phi(t-\tau) = e^{A(t-\tau)}$$

or

$$x(t) = \phi(t)X(0) + \int_0^t \phi(t-\tau)B\,u(\tau)\,d\tau$$

$$\phi(s) = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}^{-1} = \frac{1}{\Delta}\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

where $\Delta = s^2 + 3s + 2 = (s+1)(s+2)$

$$X(s) = \phi(s)X(0) = \frac{1}{\Delta}\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} s+4 \\ s-2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{s+4}{(s+1)(s+2)} \\ \frac{s-2}{(s+1)(s+2)} \end{bmatrix} = \begin{bmatrix} \frac{3}{s+1} + \frac{-2}{s+2} \\ \frac{-3}{s+1} + \frac{4}{s+2} \end{bmatrix}$$

$$x(t) = \mathcal{L}^{-1}X(s) = \begin{bmatrix} 3e^{-t} - 2e^{-2t} \\ -3e^{-t} + 4e^{-2t} \end{bmatrix}$$

$$y(t) = 6e^{-t} - 4e^{-2t}$$

Q2 For the system given below, obtain,
(a) zero input response. (b) zero state response.
(c) Total response (d) Output response.

$$\dot{x} = Ax + Bu \qquad A = \begin{bmatrix} 1 & 4 \\ -2 & -5 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad u = 1$$

$$y = Cx$$

$$x_1(0) = 1$$
$$x_2(0) = 0$$

Solution →

$$x(t) = \phi(t)\, x(0) + \int_0^t \phi(t-\tau) B u(\tau)\, d\tau$$

$$\phi(t) = \mathcal{L}^{-1}\phi(s), \qquad \phi(s) = (sI - A)^{-1}$$

$$= \begin{bmatrix} s-1 & -4 \\ 2 & s+5 \end{bmatrix}^{-1} = \frac{1}{D}\begin{bmatrix} s+5 & 4 \\ -2 & s-1 \end{bmatrix}$$

$$D = s^2 + 4s - 5 + 8 = s^2 + 4s + 3 = (s+3)(s+1)$$

$$x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\phi(t)x(0) = \mathcal{L}^{-1}\left\{ \frac{1}{D}\begin{bmatrix} s+5 & 4 \\ -2 & s-1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$

$$= \mathcal{L}^{-1}\left\{ \frac{1}{D}\begin{bmatrix} s+5 \\ -2 \end{bmatrix} \right\} = \mathcal{L}^{-1}\left\{ \begin{bmatrix} \frac{-1}{s+3} + \frac{2}{s+1} \\ \frac{1}{s+3} + \frac{-1}{s+1} \end{bmatrix} \right\}$$

(a) Zero Input response :→

$$x(t)\big|_{ZIR} \quad \phi(t)x(0) = \begin{bmatrix} -1e^{-3t} + 2e^{-t} \\ e^{-3t} - e^{-t} \end{bmatrix} = \begin{bmatrix} 2e^{-t} - e^{-3t} \\ -e^{-t} + e^{-3t} \end{bmatrix}$$

(b) Zero State response :→

$$x(t)\Big|_{ZSR} = \mathcal{L}^{-1}\left\{ \phi(s)\, B\, U(s) \right\} = \frac{1}{D}\begin{bmatrix} s+5 & 4 \\ -2 & s+1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\left(\frac{1}{s}\right)$$

$$= \mathcal{L}^{-1}\left\{ \frac{1}{D}\begin{bmatrix} \frac{4}{s} \\ \frac{s+1}{s} \end{bmatrix} \right\} = \mathcal{L}^{-1}\begin{bmatrix} \frac{4}{s(s+3)(s+1)} \\ \frac{s+1}{s(s+3)(s+1)} \end{bmatrix} = \begin{bmatrix} \frac{4}{3} - 2e^{-t} + \frac{2}{3}e^{-3t} \\ \frac{1}{3} + e^{-t} - \frac{2}{3}e^{-3t} \end{bmatrix}$$

Total response

$$x(t) = x(t)\Big|_{ZIR} + x(t)\Big|_{ZSR}$$

$$x_1(t) = (2e^{-t} - e^{-3t}) + (4/3 - 2e^{-t} + 2/3 e^{-3t}) = (4/3 - 1/3 e^{-3t})$$

$$x_2(t) = (-e^{-t} + e^{-3t}) + \left(-\frac{1}{3} + e^{-t} - \frac{2}{3}e^{-3t}\right) = \left(-\frac{1}{3} + \frac{1}{3}e^{-3t}\right)$$

(d) The o/p response,

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \left(\frac{4}{3} - \frac{1}{3}e^{-3t}\right)$$

(b) Zero state response.

$$x(t)\Big|_{ZSR} = \mathcal{L}^{-1}\left\{\phi(s) B U(s)\right\} = \int_0^t \phi(t-\tau) B u(\tau) d\tau$$

$$\phi(t) = \mathcal{L}^{-1}\phi(s) = \mathcal{L}^{-1}\left\{\frac{1}{D}\begin{bmatrix} s+5 & 4 \\ -2 & s-1 \end{bmatrix}\right\}$$

$$= \mathcal{L}^{-1}\begin{bmatrix} \frac{(s+5)}{(s+3)(s+2)} & \frac{4}{(s+3)(s+2)} \\ \frac{-2}{(s+3)(s+2)} & \frac{(s-1)}{(s+3)(s+2)} \end{bmatrix} = \begin{bmatrix} \frac{-2}{s+3} + \frac{3}{s+2} & \frac{4\times-1}{s+3} + \frac{4\times1}{s+2} \\ \frac{-2\times-1}{s+3} + \frac{-2\times1}{s+2} & \frac{4}{s+3} + \frac{-3}{s+2} \end{bmatrix}$$

$$= \begin{bmatrix} -2e^{-3t} + 3e^{-2t} & -4e^{-3t} + 4e^{-2t} \\ 2e^{-3t} - 2e^{-2t} & 4e^{-3t} - 3e^{-2t} \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\int_0^t \phi(t-\tau) B u(\tau) d\tau = \begin{bmatrix} -4e^{-3(t-\tau)} + 4e^{-2(t-\tau)} \\ 4e^{-3(t-\tau)} - 3e^{-2(t-\tau)} \end{bmatrix}$$

$$I_{11} = \int_0^t -4e^{-3(t-\tau)} d\tau = \int_0^t \left(-4e^{-3t} \cdot e^{3\tau} + 4e^{-2t} \cdot e^{2\tau}\right) d\tau$$

$$= -\frac{4}{3}e^{-3t}\left[e^{3\tau}\right]_0^t = -\frac{4}{3}e^{-3t}\left[e^{3t} - 1\right] + 2e^{-2t}\left[e^{2t} - 1\right]$$

$$I_{11} = \int_0^t -4e^{-3(t-\tau)}\,d\tau = \int_0^t \left(-4e^{-3t}\cdot e^{3\tau} + 4e^{-2t}\cdot e^{2\tau}\right) d\tau$$

$$= \frac{-4}{3}e^{-3t}\left[e^{3\tau}\right]_0^t = \frac{-4}{3}e^{-3t}\left(e^{3t}-1\right) + 2e^{-2t}\left[e^{2t}-1\right]$$

$$= \frac{-4}{3} + \frac{4}{3}e^{-3t} + 2 - 2e^{-2t}$$

(2) Zero state Input ;→

$$\phi(t) = \mathcal{L}^{-1}\phi(s) = \mathcal{L}^{-1}\left\{\frac{1}{\Delta}\begin{bmatrix} s+5 & 4 \\ -2 & s-1 \end{bmatrix}\right\}$$

$$= \mathcal{L}^{-1}\begin{bmatrix} \frac{s+5}{(s+3)(s+1)} & \frac{4}{(s+3)(s+1)} \\ \frac{-2}{(s+3)(s+1)} & \frac{s-1}{(s+3)(s+1)} \end{bmatrix} = \begin{bmatrix} \frac{-1}{s+3} + \frac{2}{s+1} & \frac{4\times\frac{-1}{2}}{s+3} + \frac{4\ldots}{s\ldots} \\ \frac{-2\times\frac{-1}{2}}{s+3} + \frac{-2\times\frac{1}{2}}{s+1} & \frac{2}{s+3} + \frac{-\ldots}{s+\ldots} \end{bmatrix}$$

$$\phi(t) = \begin{bmatrix} -e^{-3t} + 2e^{-t} & -2e^{-3t} + 2e^{-t} \\ e^{-3t} - e^{-t} & 2e^{-3t} - e^{-t} \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\int_0^t \phi(t-\tau)\, B\, u(\tau)\, d\tau = \begin{bmatrix} \int_0^t -2e^{-3(t-\tau)} + 2e^{-(t-\tau)}\, d\tau \\ \int_0^t 2e^{-3(t-\tau)} - e^{-(t-\tau)}\, d\tau \end{bmatrix}$$

$$= \begin{bmatrix} \int_0^t -2e^{-3t}\cdot e^{3\tau} + 2e^{-t}\cdot e^{\tau}\, d\tau \\ \int_0^t 2e^{-3t}\cdot e^{3\tau} - e^{-t}\cdot e^{\tau}\, d\tau \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-2}{3} - \frac{-2}{3}e^{-3t} + 2 - 2e^{-t} \\ \frac{2}{3} - \frac{2}{3}e^{-3t} - 1 + e^{-t} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{2}{3} - \left(-\frac{2}{3}\right)e^{-3t} + 2 - 2e^{-t} \\ \frac{2}{3} - \frac{2}{3}e^{-3t} - 1 + e^{-t} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4}{3} - 2e^{-t} + \frac{2}{3}e^{-3t} \\ -\frac{1}{3} + e^{-t} - \frac{2}{3}e^{-3t} \end{bmatrix}$$

Lecturer: 9

: Computation by Techniques based on the Cayley – Hamilton Theorem

Cayley - Hamilton Theorem :-

Statement of Cayley - Hamilton Theorem :- Every square matrix satisfies its own characteristic equation. This theorem provides a simple procedure for evaluating the functions of a matrix f(A) and STM in both continuous time and discrete time.

Proof can be derived from Fadeeva algorithm, Using Fadeeva algorithm sequence, the matrix B_i can also be expressed as,

$$B_1 = I_n$$

$$B_2 = AB_1 + a_1 I_n = A + a_1 I_n$$

$$B_3 = AB_2 + a_2 I_n = A^2 + a_1 A + a_2 I_n$$

$$B_4 = AB_3 + a_3 I_n = A^3 + a_1 A^2 + a_2 A + a_3 I_n$$

$$\vdots$$

$$B_n = AB_{n-1} + a_{n-1} I_n = A^{n-1} + a_1 A^{n-2} + a_2 A^{n-3} + \cdots + a_{n-1} I_n$$

Also, $AB_n + a_n I_n = 0 \quad \text{--- (1)}$

$$\therefore \quad AB_n = A^n + a_1 A^{n-1} + a_2 A^{n-2} + a_3 A^{n-3} + \cdots + a_{n-1} A$$

From (1), we have, $\boxed{A^n + a_1 A^{n-1} + a_2 A^{n-2} + a_3 A^{n-3} + \cdots + a_{n-1} A + a_n I_n = 0}$

Let $D(A) = A^n + a_1 A^{n-1} + a_2 A^{n-2} + \cdots + a_{n-1} A + a_n I_n$

This equation is the same as the characteristic equation.

$$D(s) = |sI - A| = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n = 0$$

with s^i replaced by the matrix A^i, $i = 1, 2, \ldots n$.

And hence proved.

Evaluation of the functions of a matrix $f(A)$:-

Let us express the characteristic polynomial in terms of λ,

$$D(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_{n-1}\lambda + a_n$$

with n distinct eigen values $\lambda_1, \lambda_2 \ldots \lambda_n$ of matrix A, such that

$D(\lambda_i) = 0$ for $i = 1, 2, 3, \ldots n$.

Further, the matrix A satisfies the characteristic eq^n. As per Cayley Hamilton theorem, the matrix polynomial $f(A)$ of degree higher than the order of A.

$$f(A) = \alpha_0 I + \alpha_1 A + \alpha_2 A^2 + \cdots + \alpha_n A^n + \alpha_{n+1} A^{n+1} + \cdots$$

can be computed on the basis of the scalar polynomial $f(\lambda)$.

$$f(\lambda) = \alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2 + \cdots + \alpha_n\lambda^n + \alpha_{n+1}\lambda^{n+1} + \cdots$$

Dividing $f(\lambda)$ by $D(\lambda)$, we have,

$$\frac{f(\lambda)}{D(\lambda)} = q(\lambda) + \frac{p(\lambda)}{D(\lambda)}$$

or $$f(\lambda) = q(\lambda)\, D(\lambda) + p(\lambda)$$

where, the remainder polynomial,

$$p(\lambda) = \beta_0 + \beta_1\lambda + \beta_2\lambda^2 + \cdots + \beta_{n-1}\lambda^{n-1}$$

Since, $D(\lambda_i) = 0$, $f(\lambda_i) = p(\lambda_i)$ for $i = 1, 2, \ldots n$.

The coefficients $\beta_0, \beta_1 \ldots \beta_{n-1}$ can be computed by substituting the values of $\lambda_1, \lambda_2 \ldots \lambda_n$. Substituting A for λ, we obtain

$$f(A) = q(A)\, D(A) + p(A)$$

Since, $D(A) = 0$, we obtained the desired matrix

$$f(A) = p(A) = \beta_0 I + \beta_1 A + \beta_2 A^2 + \cdots + \beta_{n-1} A^{n-1}$$

Lecturer: 10

: Computation by Techniques based on the Sylvester's Expansion theorem

Sylvester's expansion theorem:

Use numerical technique in time domain for calculating the STM in closed form. De Russo, Roy, Close [6] or ~~Zeda~~ Zadeh and Desoer [8] for proof

According to this theorem $f(A)$, the polynomial function of A,

$$f(A) = \sum_{k=0}^{\infty} g_k A^k$$

may be expressed in terms of distinct eigen values λ_i of A, as

$$f(A) = \sum_{i=1}^{n} f(\lambda_i) F_i$$

$$F_i = \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{A - \lambda_j I}{\lambda_i - \lambda_j} \quad \text{--- (1)}$$

Using Sylvester's expansion theorem STM can be determined as.

$$\phi(A) = e^{At} = \sum_{i=0}^{\infty} \frac{A^i}{i!} t^i = f(A) = \sum_{i=1}^{n} e^{\lambda_i t} F_i \quad \text{--- (2)}$$

Step for Determination of STM, :-

(1) Determine the eigen values λ_i's of A.

(2) Determine F_i's from eq^n (1)

(3) Determine the STM ~~is of the form~~. $\phi(A) = e^{At}$.

Note: Each element of the STM is of the form.

$$\phi_{ij}(t) = a_1 e^{\lambda_1 t} + a_2 e^{\lambda_2 t} + \dots + a_n e^{\lambda_n t} = \sum_{k=1}^{n} a_k e^{\lambda_k t}$$

Ex Compute STM for the system

$$\dot{x}(t) = \begin{bmatrix} -2 & 2 \\ 0 & -4 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$

Soln For this system, eign value is determined by,

$$D(\lambda) = |\lambda I - A| = \begin{vmatrix} \lambda+2 & -2 \\ 0 & \lambda+4 \end{vmatrix} = \lambda^2 + 6\lambda + 8 = 0$$

$$\therefore \lambda = -2, -4 \quad \text{or} \quad \lambda_1 = -2, \quad \lambda_2 = -4$$

The matrix F_1 and F_2 are computed as,

$$F_1 = \frac{A - \lambda_2 I}{\lambda_1 - \lambda_2} = \frac{A + 4I}{-2+4} = \left\{ \begin{bmatrix} -2 & 2 \\ 0 & -4 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \right\} \frac{1}{2}$$

$$F_1 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$F_2 = \frac{A - \lambda_1 I}{\lambda_2 - \lambda_1} = \frac{A + 2I}{-2} = \left\{ \begin{bmatrix} -2 & 2 \\ 0 & -4 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \right\} \frac{1}{-2}$$

$$F_2 = \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}$$

$$\phi(t) = \sum_{i=1}^{n} e^{\lambda_i t} F_i = e^{-2t} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + e^{-4t} \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} e^{-2t} & e^{-2t} - e^{-4t} \\ 0 & e^{-4t} \end{bmatrix}$$

Lecturer: 11

: Concept of Controllability and Observability

CONCEPTS OF CONTROLLABILITY AND OBSERVABILITY

Controllability

A system is said to be controllable if it is possible to transfer the system state from any initial state x(t_0) to any desired state x(t) in specified finite time by a control vector u(t).

A system is completely controllable if all the state variables are controllable.

Controllability criterion

Controllability is a property of the linkage between the input and the state & thus involves the matrices A and B.

For an n^{th} order linear time invariant system, controllable can be determined from the matrix shown below

Q_c =[$B \quad AB \quad A^2B \quad \ldots\ldots\ldots \quad A^{n-1}B$] is of rank n.

This test for controllability is called Kalman's test.

Observability

A system is said to be observable if each state x(t) could be determined from measurement of the output over a finite interval of time 0≤ t ≤ t_f .

Observability criterion

Observability is a property of linkage between the output & the state & thus involves the matrices A and C.

For an n^{th} order linear time invariant system is completely observable if and only if the observability matrix

Q_o = [$C^T \vdots A^TC^T \vdots A^{T2}C^T \vdots \cdots \vdots A^{T(n-1)}C^T$] is of rank n.

This is Kalman's test for observability.

Solved Problems

1) A dynamic system S represented by the state equation

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

Check whether the system is completely controllable.

Ans:- $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$AB = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$, $A^2B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 7 \end{bmatrix}$

The controllability matrix is

$Q_c = [B \quad AB \quad A^2B] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & -3 & 7 \end{bmatrix}$ and the determinant of $Q_c \neq 0$ and the rank of matrix is 3. Hence the system is completely controllable.

2) The state equation for a system is

$\dot{x} = \begin{bmatrix} -3 & -1 \\ 2 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ -1 \end{bmatrix} r$

Check whether the system is completely controllable.

Ans:- $A = \begin{bmatrix} -3 & -1 \\ 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$AB = \begin{bmatrix} -3 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$

The controllability matrix

$Q_c = [B \quad AB] = \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix}$ and the determinant of $Q_c = 0$ and the rank of matrix is 1. Hence the system is not completely controllable.

3) A linear system is described by a state model

$\dot{x} = \begin{bmatrix} -3 & -1 \\ 2 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ -1 \end{bmatrix} r$ and $y = [1 \quad -1] x$

Determine whether the system is completely observable.

Ans:- $A = \begin{bmatrix} -3 & -1 \\ 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $C^T = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$A^TC^T = \begin{bmatrix} -3 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$

The observability matrix

$Q_o = [C^T \,\vdots\, A^TC^T] = \begin{bmatrix} 1 & -5 \\ -1 & -1 \end{bmatrix}$ and determinant of $Q_o \neq 0$ and the rank of matrix is 2. Hence the system is completely observable.

Lecturer: 12

: Effect of Pole-zero Cancellation in Transfer Function

POLE-ZERO CANCELLATIONS AND STABILITY

Consider the linear time-invariant system given by the transfer function.

$$H(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} = \frac{N(s)}{D(s)} \ .$$

Recall that this system is stable if all of the poles are in the OLHP, and these poles are the roots of the polynomial D(s). It is important to note that one should not cancel any common poles and zeros of the transfer function before checking the roots of D (s). Specifically, suppose that both of the polynomials N(s) and D (s) have a roots =a f or some complex (or real)number a. One must not cancel out this common zero and pole in the transfer function before testing for stability. The reason for this is that, even though the pole will not show up in the response to the input, it will still appear as a result of any initial conditions in the system, or due to additional inputs entering the system (such as disturbances). If the pole and zero are in the CRHP, the system response might blow up due to these

Initial conditions or disturbances, even though the input to the system Is bounded, and this would violate BIBO stability

To see this a little more clearly, consider the following example. Suppose the transfer function of a linear system is given by

$$H(s) = \frac{s-1}{s^2+2s-3} = \frac{N(s)}{D(s)}$$

Nothing that $s^2+2s-3=(s+3)(s-1)$ suppose we decided to cancel out the common pole and zero at s=0 to obtain

$$H(s) = \frac{1}{s+3} \ .$$

Based on this transfer function we might conclude that the system is stable, since it only has a pole in OLHP . what we should actually dois look at the original denominator D(s) , and correctly conclude that the system is un stable because one of the pole in the CRHP. To see why the pole zero cancellation hide the stability of the system first writes out the differential equation corresponding to the transfer function we obtain

$$\ddot{y} + 2\dot{y} - 3y = \dot{u} - u \ .$$

Taking Laplace transformation on both side and taking initial condition in count

$$s^2Y(s) - sy(0) - \dot{y}(0) + 2sY(s) - 2y(0) - 3Y(s) = sU(s) - u(0) - U(s) \ .$$

Rearranging the equation we get

$$Y(s) = \underbrace{\frac{s-1}{s^2+2s-3}}_{H(s)}U(s) + \frac{s+2}{s^2+2s-3}y(0) + \frac{1}{s^2+2s-3}\dot{y}(0) - \frac{1}{s^2+2s-3}u(0) \ .$$

Note that the denominator polynomial in each of the terms on the right hand sides is equal to D(s) (the denominator of the transfer function). For simplicity, suppose that $y(0)=y_0$ (for some real number y_0) $\dot{y}(0) = 0$ and $u(0) = 0$.The partial fraction expansion of the term $\frac{s+2}{s^2+2s-3}y_0$ is given by

$$\frac{s+2}{s^2+2s-3}y_0 = \frac{y_0}{4}\left(\frac{1}{s+3} + \frac{1}{s-1}\right)$$

and this contributes the term $\frac{y_0}{4}\left(e^{-3t} + e^{t}\right), t \geq 0,$ to the response of the system. Note that the e^t term blows up, and thus the output of the system blows up if y_0 is not zero, even if the input to the system is bounded.

MODULE-3

Lecture-1
Behaviour of Non linear Systems, Investigation of nonlinear systems

Behaviour of Non linear Systems:

The most important feature of nonlinear systems is that nonlinear systems do not obey the principle of superposition. Due to this reason, in contrast to the linear case, the response of nonlinear systems to a particular test signal is no guide to their behaviour to other inputs. The nonlinear system response may be highly sensitive to input amplitude. For example, a nonlinear system giving best response for a certain step input may exhibit highly unsatisfactory behaviour when the input amplitude is changed. Hence, in a nonlinear system, the stability is very much dependent on the input and also the initial state.

Further, the nonlinear systems may exhibit limit cycles which are self-sustained oscillations of fixed frequency and amplitude. Once the system trajectories converge to a limit cycle, it will continue to remain in the closed trajectory in the state space identified as limit cycles. In many systems the limit cycles are undesirable particularly when the amplitude is not small and result in some unwanted phenomena.

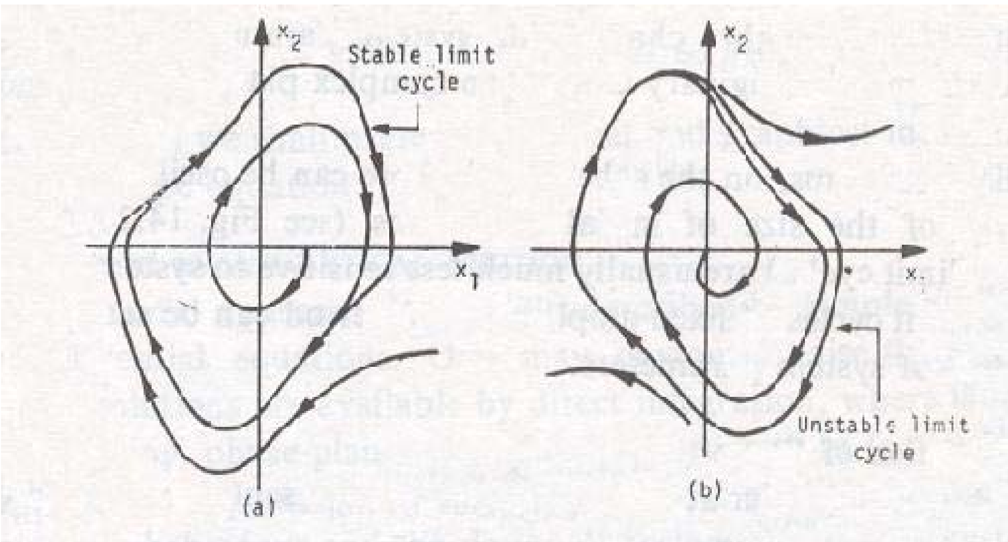
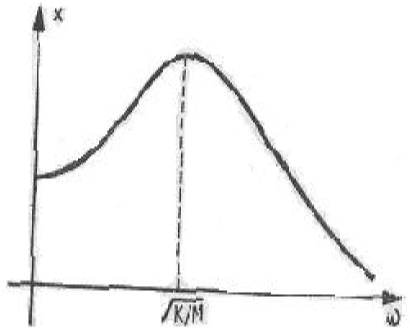


Figure 6.1

A nonlinear system, when excited by a sinusoidal input, may generate several harmonics in addition to the fundamental corresponding to the input frequency. The amplitude of the fundamental is usually the largest, but the harmonics may be of significant amplitude in many situations.

Another peculiar characteristic exhibited by nonlinear systems is called jump phenomenon. For example, let us consider the frequency response curve of spring-mass damper system. The frequency responses of the system with a linear spring, hard spring and soft spring are as shown in Fig. 6.2(a), Fig. 6.2(b) and Fig. 6.2(c) respectively. For a hard spring, as the input frequency is gradually increased from zero, the measured response follows the curve through the A, B and C, but at C an increment in frequency results in discontinuous jump down to the point D, after which with further increase in frequency, the response curve follows through DE. If the frequency is now decreased, the response follows the curve EDF with a jump up to B from the point F and then the response curve moves towards A. This phenomenon which is peculiar to nonlinear systems is known as jump resonance. For a soft spring, jump phenomenon will happen as shown in fig. 6.2(c).



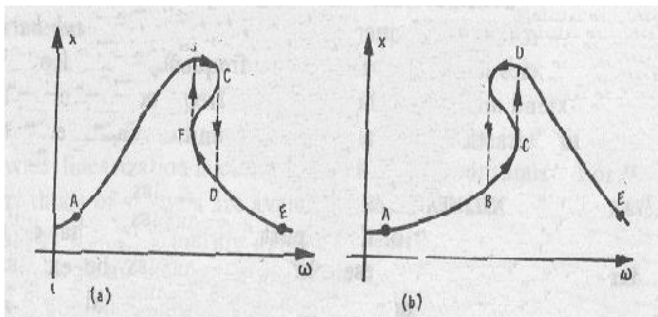


Fig. 6.2(a) Fig. 6.2(b) Fig. 6.2 (c)

When excited by a sinusoidal input of constant frequency and the amplitude is increased from low values, the output frequency at some point exactly matches with the input frequency and continue to remain as such thereafter. This phenomenon which results in a synchronization or matching of the output frequency with the input frequency is called ***frequency entrainment*** or synchronization.

Classification of Nonlinearities:

The nonlinearities are classified into i) Inherent nonlinearities and ii) Intentional nonlinearities.

The nonlinearities which are present in the components used in system due to the inherent imperfections or properties of the system are known as inherent nonlinearities. Examples are saturation in magnetic circuits, dead zone, back lash in gears etc. However in some cases introduction of nonlinearity may improve the performance of the system, make the system more economical consuming less space and more reliable than the linear system designed to achieve the same objective. Such nonlinearities introduced intentionally to improve the system performance are known as intentional nonlinearities. Examples are different types of relays which are very frequently used to perform various tasks. But it should be noted that the improvement in system performance due to nonlinearity is possible only under specific operating conditions. For other conditions, generally nonlinearity degrades the performance of the system.

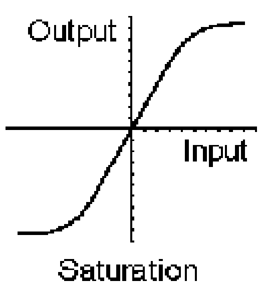
Lecture-2
Saturation, Friction, Backlash, Relay, Multivariable Nonlinearity.

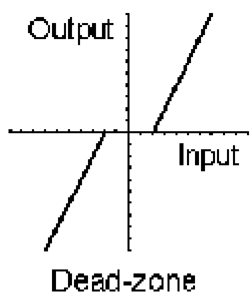
Common Physical Non Linearities:

The common examples of physical nonlinearities are saturation, dead zone, coulomb friction, stiction, backlash, different types of springs, different types of relays etc.

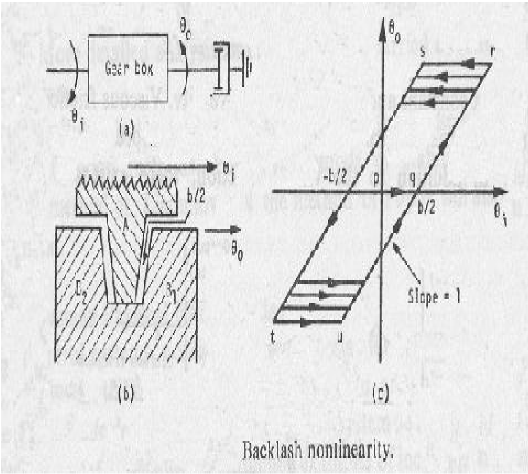
Saturation: This is the most common of all nonlinearities. All practical systems, when driven by sufficiently large signals, exhibit the phenomenon of saturation due to limitations of physical capabilities of their components. Saturation is a common phenomenon in magnetic circuits and amplifiers.

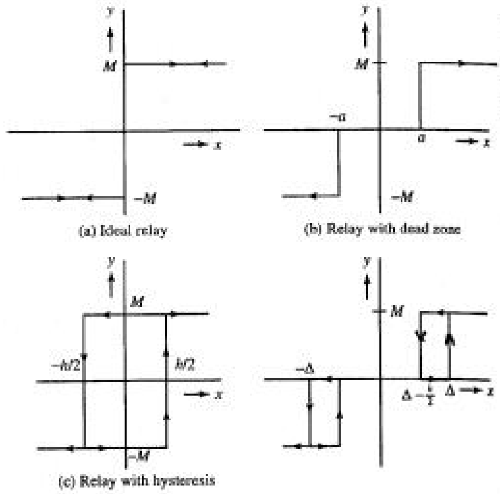
Dead zone: Some systems do not respond to very small input signals. For a particular range of input, the output is zero. This is called dead zone existing in a system. The input-output curve is shown in figure.





Backlash: Another important nonlinearity commonly occurring in physical systems is hysteresis in mechanical transmission such as gear trains and linkages. This nonlinearity is somewhat different from magnetic hysteresis and is commonly referred to as backlash. In servo systems, the gear backlash may cause sustained oscillations or chattering phenomenon and the system may even turn unstable for large backlash.





Relay: A relay is a nonlinear power amplifier which can provide large power amplification inexpensively and is therefore deliberately introduced in control systems. A relay controlled system can be switched abruptly between several discrete states which are usually off, full forward and full reverse. Relay controlled systems find wide applications in the control field. The characteristic of an ideal relay is as shown in figure. In practice a relay has a definite amount of dead zone as shown. This dead zone is caused by the facts that relay coil requires a finite amount of current to actuate the relay. Further, since a larger coil current is needed to close the relay than the current at which the relay drops out, the characteristic always exhibits hysteresis.

Multivariable Nonlinearity: Some nonlinearities such as the torque-speed characteristics of a servomotor, transistor characteristics etc., are functions of more than one variable. Such nonlinearities are called multivariable nonlinearities.

Lecture-3,4
Basic Concepts, Singular Points: Nodal Point, Saddle Point, Focus Point, Centre or Vortex Point

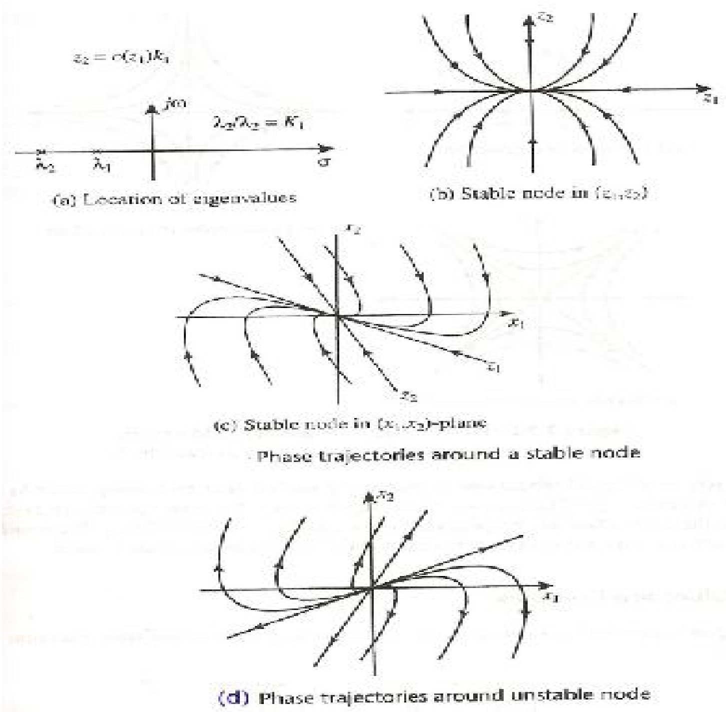
The Phase Plane Method:

This method is applicable to second order linear or nonlinear systems for the study of the nature of phase trajectories near the equilibrium points. The system behaviour is qualitatively analysed along with design of system parameters so as to get the desired response from the system. The periodic oscillations in nonlinear systems called limit cycle can be identified with this method which helps in investigating the stability of the system.

Singular points: From the fundamental theorem of uniqueness of solutions of the state equations or differential equations, it can be seen that the solution of the state equation starting from an initial state in the state space is unique. This will be true if $f_1(x_1, x_2)$ and $f_2(x_1,x_2)$ are analytic. For such a system, consider the points in the state space at which the derivatives of all the state variables are zero. These points are called ***singular points***. These are in fact ***equilibrium points*** of the system. If the system is placed at such a point, it will continue to lie there if left undisturbed. A family of phase trajectories starting from different initial states is called a phase portrait. As time t increases, the phase portrait graphically shows how the system moves in the entire state plane from the initial states in the different regions. Since the solutions from each of the initial conditions are unique, the phase trajectories do not cross one another. If the system has nonlinear elements which are piece-wise linear, the complete state space can be divided into different regions and phase plane trajectories constructed for each of the regions separately.

Nodal Point: Consider eigen values are real, distinct and negative as shown in figure (a). For this case the equation of the phase trajectory follows as $z_2 = c\,(z_1)^{k_1}$

Where k1= $\left(\frac{\lambda_2}{\lambda_1}\right) \geq 0$ so that the trajectories become a set of parabola as shown in figure (b) and the equilibrium point is called a node. In the original system of coordinates, these trajectories appear to be skewed as shown in figure (c).



(a) Location of eigenvalues

(b) Stable node in (z_1, z_2)

(c) Stable node in (x_1, x_2)-plane

Phase trajectories around a stable node

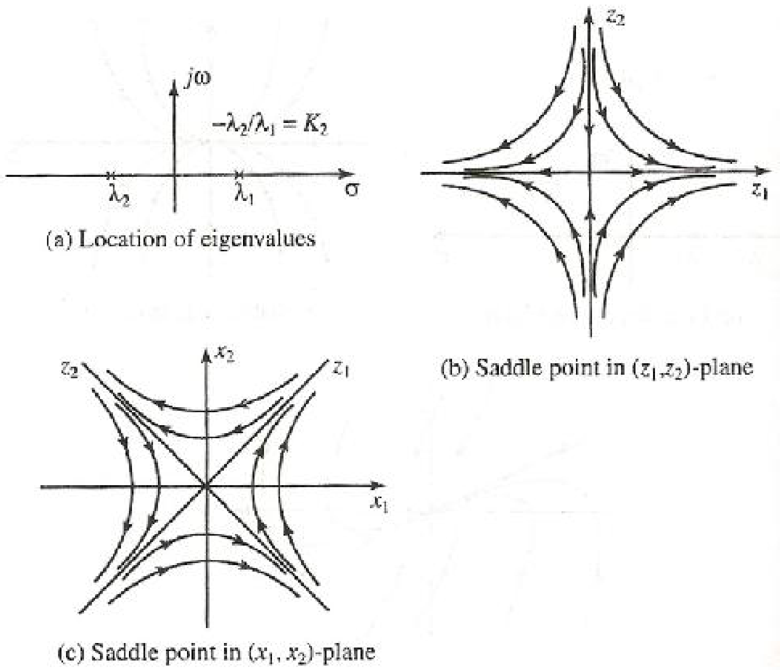
(d) Phase trajectories around unstable node

If the eigen values are both positive, the nature of the trajectories does not change, except that the trajectories diverge out from the equilibrium point as both $z_1(t)$ and $z_2(t)$ are increasing exponentially. The phase trajectories in the *x1-x2* plane are as shown in figure (d). This type of singularity is identified as a node, but it is an unstable node as the trajectories diverge from the equilibrium point.

Where $z_1(t)=e^{\lambda_1 t} z_1(0)$ $z_2(t)= e^{\lambda_2 t} z_2(0)$

Saddle Point:

Consider now a system with eigen values are real, distinct one positive and one negative. Here, one of the states corresponding to the negative eigen value converges and the one corresponding to positive eigen value diverges so that the trajectories are given by $z_2 = c(z_1)^{-k}$ or $(z_1)^k z_2 = c$ which is an equation to a rectangular hyperbola for positive values of k. The location of the eigen values, the phase portrait in $z_1 - z_2$ plane and in the $x_1 - x_2$ plane are as shown in figure. The equilibrium point around which the trajectories are of this type is called a saddle point



Phase trajectories around saddle point

Focus Point:

Consider a system with complex conjugate eigen values. The canonical form of the state equation can be written as

$$\begin{bmatrix} z_1' \\ z_2' \end{bmatrix} = \begin{bmatrix} \sigma + j\omega & 0 \\ 0 & \sigma - j\omega \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

Using linear transformation, the equation becomes

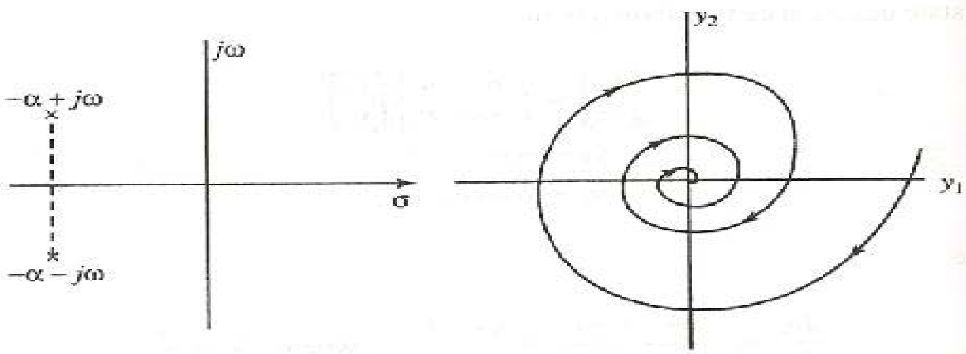
$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

The slope $\frac{dy_2}{dy_1} = \frac{-\omega y_1 + \sigma y_2}{\sigma y_1 + \omega y_2} = \frac{y_2 - k y_1}{y_1 - k y_2}$

Define $\frac{dy_2}{dy_1} = \tan\psi$, $\frac{y_2}{y_1} = \tan\theta$

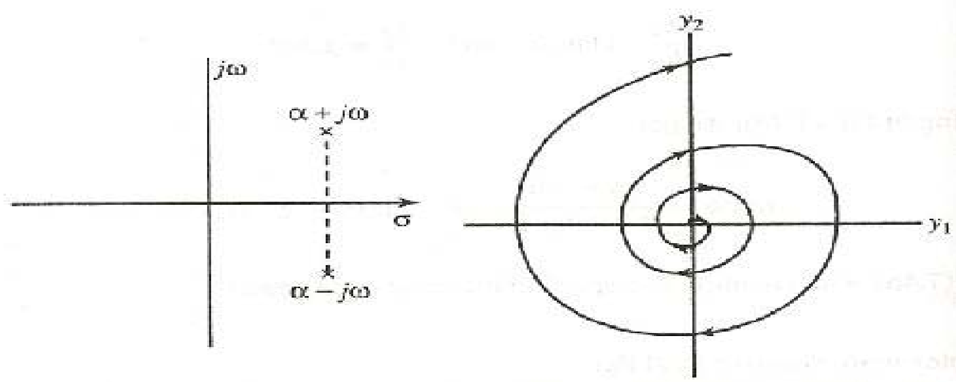
We get $\tan\psi = \dfrac{\tan\theta - K}{1 + K\tan\theta}$ or $\tan(\theta - \psi) = k$

This is an equation for a spiral in the polar coordinates. A plot of this equation for negative values of real part is a family of equiangular spirals. The origin which is a singular point in this case is called a *stable focus*. When the eigen values are complex conjugate with positive real parts, the phase portrait consists of expanding spirals as shown in figure and the singular point is an *unstable focus*. When transformed into the x_1-x_2 plane, the phase portrait in the above two cases is essentially spiralling in nature, except that the spirals are now somewhat twisted in shape.



(a) Eigen values (b) Phase transectories-stable focus

Phase trajectories around stable focus



(a) Eigen values (b) Phase trajectories unstable focus

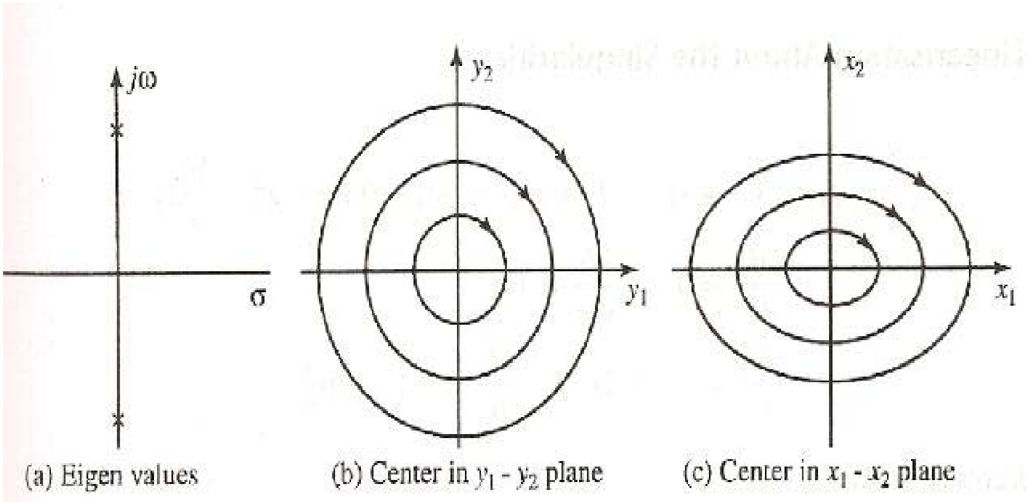
Phase trajectories around unstable focus

Centre or Vortex Point:

Consider now the case of complex conjugate eigen values with zero real parts.
ie., $\lambda_1, \lambda_2 = \pm j\omega$

$$\frac{dy_2}{dy_1} = \frac{j\omega y_1}{-j\omega y_2} = -\frac{y_1}{y_2} \text{ from which } y_1\, dy_1 + y_2\, dy_2 = 0$$

Integrating the above equation, we get $y_1^2 + y_2^2 = R^2$ which is an equation to a circle of radius R. The radius R can be evaluated from the initial conditions. The trajectories are thus concentric circles in y_1-y_2 plane and ellipses in the x_1-x_2 plane as shown in figure. Such a singular points, around which the state trajectories are concentric circles or ellipses, are called a *centre* or *vortex*.



Phase trajectories around vortex or center

Lecture-5
Stability of Non Linear Systems: Limit Cycles

Limit Cycles:

Limit cycles have a distinct geometric configuration in the phase plane portrait, namely, that of an isolated closed path in the phase plane. A given system may have more than one limit cycle. A limit cycle represents a steady state oscillation, to which or from which all trajectories nearby will converge or diverge. In a nonlinear system, limit cycles describes the amplitude and period of a self sustained oscillation. It should be pointed out that not all closed curves in the phase plane are limit cycles. A phase-plane portrait of a conservative system, in which there is no damping to dissipate energy, is a continuous family of closed curves. Closed curves of this kind are not limit cycles because none of these curves are isolated from one another. Such trajectories always occur as a continuous family, so that there are closed curves in any neighborhoods of any particular closed curve. On the other hand, limit cycles are periodic motions exhibited only by nonlinear non conservative systems.

As an example, let us consider the well known Vander Pol's differential equation

$$\frac{d^2x}{dt^2} - \mu(1 - x^2)\frac{dx}{dt} + x = 0$$

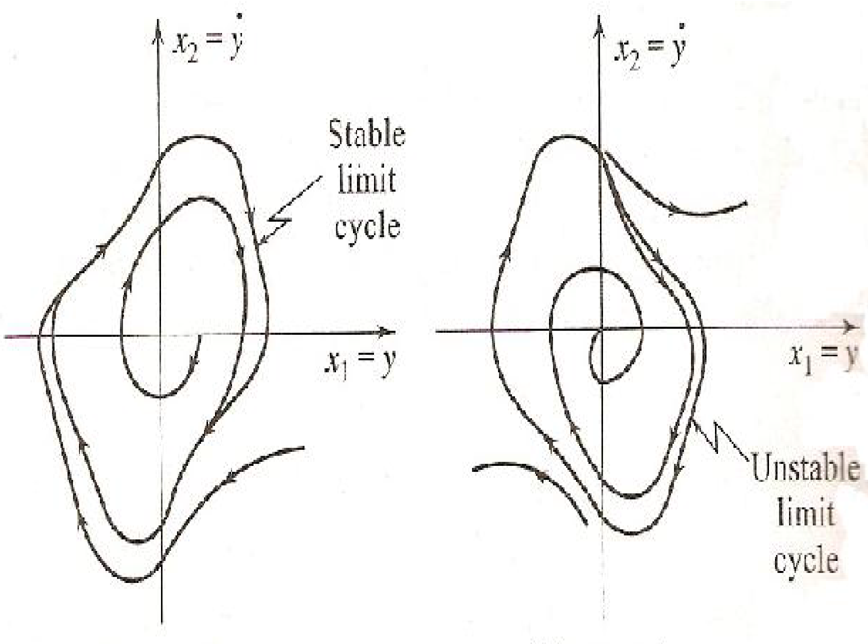
This describes physical situations in many nonlinear systems.

In terms of the state variables $x_1 = x$ and $x_2 = \dot{x}_1$ we obtained

$\dot{x}_1 = x_2$

$\dot{x}_2 = \mu(1 - x_1^2)x_2 - x_1$

The figure shows the phase trajectories of the system for μ > 0 and μ < 0. In case of μ > 0 we observe that for large values of x1(0), the system response is damped and the amplitude of x1(t) decreases till the system state enters the limit cycle as shown by the outer trajectory. On the other hand, if initially x1(0) is small, the damping is negative, and hence the amplitude of x1(t) increases till the system state enters the limit cycle as shown by the inner trajectory. When μ < 0, the trajectories moves in the opposite directions as shown in figure.



(i) $\mu > 0$ (ii) $\mu < 0$

A limit cycle is called stable if trajectories near the limit cycle, originating from outside or inside, converge to that limit cycle. In this case, the system exhibits a sustained oscillation with constant amplitude. This is shown in figure (i). The inside of the limit cycle is an unstable region in the sense that trajectories diverge to the limit cycle, and the outside is a stable region in the sense that trajectories converge to the limit cycle.

A limit cycle is called an unstable one if trajectories near it diverge from this limit cycle. In this case, an unstable region surrounds a stable region. If a trajectory starts within the stable region, it converges to a singular point within the limit cycle. If a trajectory starts in the unstable region, it diverges with time to infinity as shown in figure (ii). The inside of an unstable limit cycle is the stable region, and the outside the unstable region.

In the phase plane, *a limit cycle is defied as an isolated closed curve*. The trajectory has to be both closed, indicating the periodic nature of the motion, and isolated, indicating the limiting nature of the cycle (with near by trajectories
converging or diverging from it).

Depending on the motion patterns of the trajectories in the vicinity of the limit cycle, we can distinguish three kinds of limit cycles

- **Stable Limit Cycles**: all trajectories in the vicinity of the limit cycle converge to it as $t \to \infty$ (Fig. 2.10.a).
- **Unstable Limit Cycles**: all trajectories in the vicinity of the limit cycle diverge to it as $t \to \infty$ (Fig. 2.10.b)
- **Semi-Stable Limit Cycles**: some of the trajectories in the vicinity of the limit cycle converge to it as $t \to \infty$ (Fig. 2.10.c)

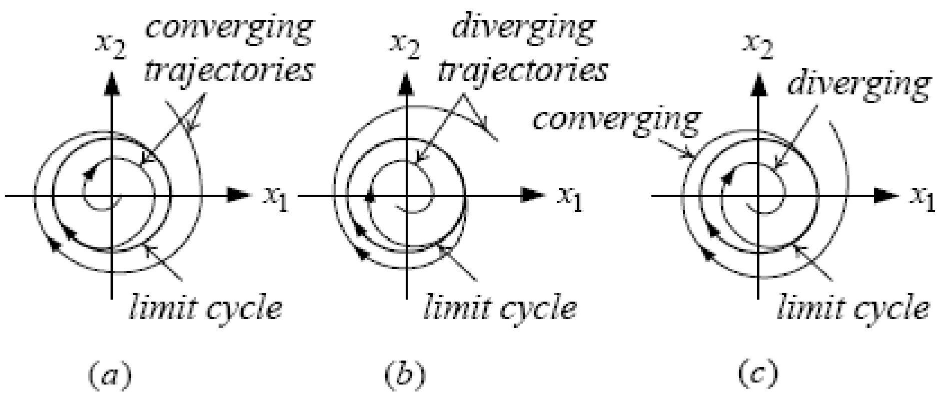


Fig. 2.10 Stable, unstable, and semi-stable limit cycles

Lecture-6
Construction by Analytical Method, Construction by Graphical Methods

Construction of Phase Trajectories:

Consider the homogenous second order system with differential equations

$$M\frac{d^2x}{dt^2}+f\frac{dx}{dt}+kx=0$$

This equation may be written in the standard form

$$\ddot{x}+2\delta\omega_n\dot{x}+\omega_n^2x=0$$

where ζ and ω_n are the damping factor and undamped natural frequency of the system.Defining the state variables as x = x_1 and $\dot{x}$ = x_2, we get the state equation in the state variable form as

$$\dot{x}=x_2$$

$$\dot{x}_2=-2\delta\omega_nx_2-\omega_n^2x_1$$

These equations may then be solved for phase variables x_1 and x_2. The time response plots of x_1, x_2 for various values of damping with initial conditions can be plotted. When the differential equations describing the dynamics of the system are nonlinear, it is in general not possible to obtain a closed form solution of x_1, x_2. For example, if the spring force is nonlinear say ($k_1 x + k_2 x^3$) the state equation takes the form

$$\dot{x}=x_2$$

$$\dot{x}_2=-\frac{f}{M}x_2-\frac{k_1}{M}x_1-\frac{k_2}{M}x_1^3$$

Solving these equations by integration is no more an easy task. In such situations, a graphical method known as the phase-plane method is found to be very helpful. The coordinate plane with axes that correspond to the dependent variable x_1 and x_2 is called phase-plane. The curve described by the state point (x_1,x_2) in the phase-plane with respect to time is called a phase trajectory. A phase trajectory can be easily constructed by graphical techniques.

Isoclines Method:

Let the state equations for a nonlinear system be in the form

$$\dot{x}_1 = f_1(x_1,x_2)$$

$$\dot{x}_2 = f_2(x_1,x_2)$$

When both $f_1(x_1,x_2)$ and $f_2(x_1,x_2)$ are analytic.

From the above equation, the slope of the trajectory is given by

$$\frac{dx_2}{dx_1} = \frac{f2(x1,x2)}{f1(x1,x2)} = M$$

Therefore, the locus of constant slope of the trajectory is given by

$$f_2(x_1,x_2) = M\, f_1(x_1,x_2)$$

The above equation gives the equation to the family of isoclines. For different values of M, the slope of the trajectory, different isoclines can be drawn in the phase plane. Knowing the value of M on a given isoclines, it is easy to draw line segments on each of these isoclines.

Consider a simple linear system with state equations

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_2 - x_1$$

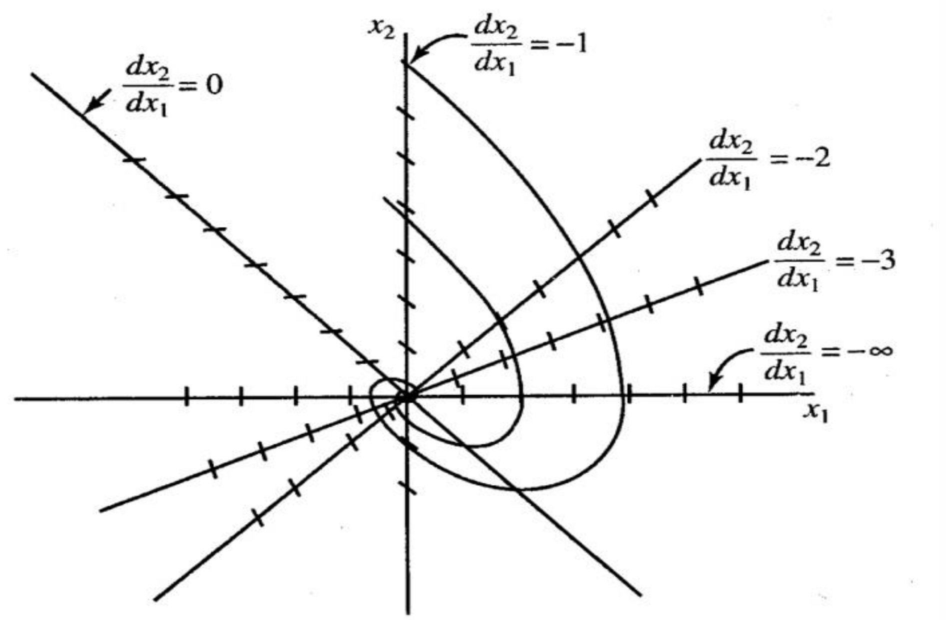
Dividing the above equations we get the slope of the state trajectory in the x_1-x_2 plane as

$$\frac{dx_2}{dx_1} = \frac{-x_2 - x_1}{x_2} = M$$

For a constant value of this slope say M, we get a set of equations

$$x_2 = \frac{-1}{M+1}x_1$$

which is a straight line in the x_1-x_2 plane. We can draw different lines in the x_1-x_2 plane for different values of M; called isoclines. If draw sufficiently large number of isoclines to cover the complete state space as shown, we can see how the state trajectories are moving in the state plane. Different trajectories can be drawn from different initial conditions. A large number of such trajectories together form a phase portrait. A few typical trajectories are shown in figure given below.



Phase plot

The Procedure for construction of the phase trajectories can be summarised as below:

1.For the given nonlinear differential equation, define the state variables as x_1 and x_2 and obtain the state equations as

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = f(x_1, x_2)$$

2.Determine the equation to the isoclines as

$$\frac{dx_2}{dx_1} = \frac{f(x_1, x_2)}{x_2} = M$$

3. For typical values of M, draw a large number of isoclines in x_1-x_2 plane

4. On each of the isoclines, draw small line segments with a slope M.

5. From an initial condition point, draw a trajectory following the line segments With slopes M on each of the isoclines.

Delta Method:

The delta method of constructing phase trajectories is applied to systems of the form

$$\ddot{x} + f(x, \dot{x}, t) = 0$$

Where $f(x, \dot{x}, t)$ may be linear or nonlinear and may even be time varying but must be continuous and single valued.

With the help of this method, phase trajectory for any system with step or ramp or any time varying input can be conveniently drawn. The method results in considerable time saving when a single or a few phase trajectories are required rather than a complete phase portrait. While applying the delta method, the above equation is first converted to the form

$$\ddot{x} + \omega_n[x + \delta(x, \dot{x}, t)] = 0$$

In general $\delta(x, \dot{x}, t)$ depends upon the variables x, $\dot{x}$ and t, but for short intervals the changes in these variables are negligible. Thus over a short interval, we have

$\ddot{x} + \omega_n[x + \delta] = 0$, where δ is a constant.

Let us choose the state variables as $x_1 = x$; $x_2 = \frac{\dot{x}}{\omega_n}$,then

$$\dot{x}_1 = \omega_n x_2$$

$$\dot{x}_2 = -\omega_n[x_1 + \delta]$$

Therefore, the slope equation over a short interval is given by

$$\frac{dx_2}{dx_1} = \frac{-x_1 + \delta}{x_2}$$

With δ known at any point P on the trajectory and assumed constant for a short interval, we can draw a short segment of the trajectory by using the trajectory slope dx_2/dx_1 given in the above equation. A simple geometrical construction given below can be used for this purpose.

1. From the initial point, calculate the value of δ.
2. Draw a short arc segment through the initial point with (-δ, 0) as centre, thereby determining a new point on the trajectory.
3. Repeat the process at the new point and continue.

Example : For the system described by the equation given below, construct the trajectory starting at the initial point (1, 0) using delta method.

$$\ddot{x} + \dot{x} + x^3 = 0$$

Let $x = x_1$, and $\dot{x} = x_2$,then

$$\dot{x}_1 = x_2$$

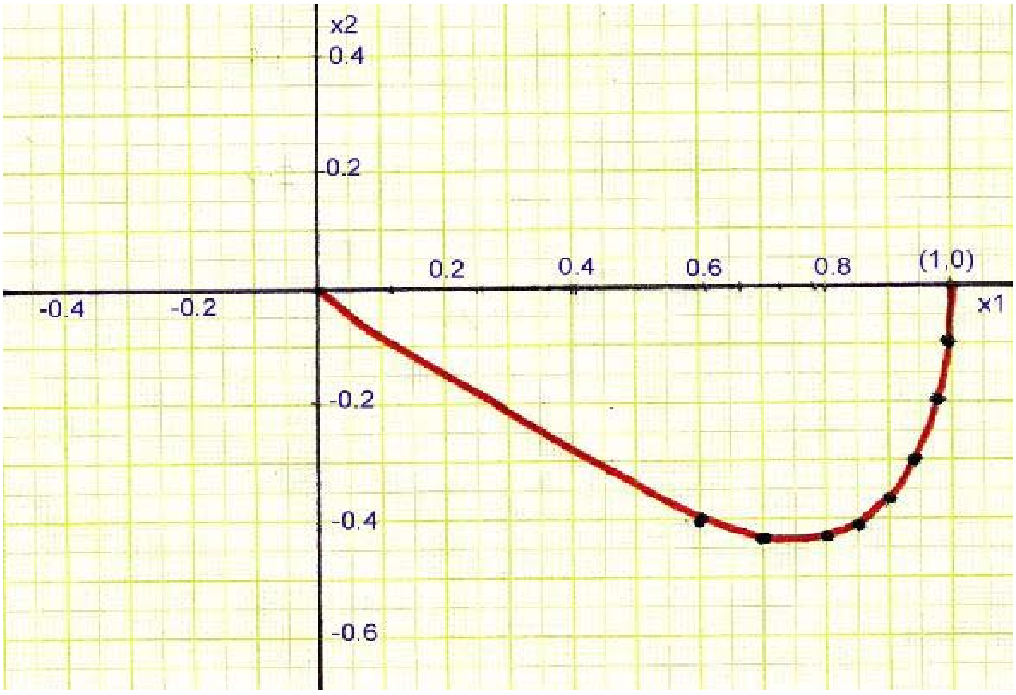
$$\dot{x}_2 = -x_2 - x_1^3$$

The above equation can be rearranged as

$$\dot{x}_2 = -(x_1 + x_2 + x_1^3 - x_1)$$

Where $\delta = x_2 + x_1^3 - x_1$

At initial point δ is calculated as δ = 0+1-1 = 0. Therefore, the initial arc is centered at point (0, 0). The mean value of the coordinates of the two ends of the arc is used to calculate the next value of δ and the procedure is continued. By constructing the small arcs in this way the complete trajectory will be obtained as shown in figure.

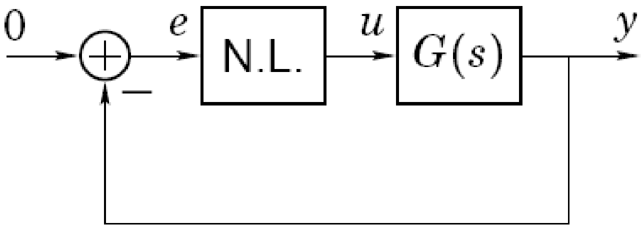


Lecture-7,8
The Describing Function Method, Derivation of Describing Functions

Describing Function

Describing functions provide a method for the analysis of nonlinear systems that is closely related to the linear-system techniques involving Bode or gain-phase plots. It is possible to use this type of analysis to determine *if limit cycles* (constant-amplitude periodic oscillations) are possible for a given system. It is also possible to use describing functions to predict the response of certain nonlinear systems to purely sinusoidal excitation, although this topic is not covered here.[2] unfortunately, since the frequency response and transient response of nonlinear systems are not directly related, the determination of transient response is not possible via describing functions.

Certain approximate techniques exist which are capable of determining the behavior of wider class of systems then is possible by phase plane method. These techniques are known as the describing function techniques .Let us consider the block diagram of a nonlinear system shown in figure, where in the blocks G(S) represent the linear elements, while the block N represents the nonlinear elements.



Let us assume that x to be non linearity is sinusoidal,

$x = X \sin \omega t$

With such an input, the output of nonlinear element will in general be non sinusoidal periodic function which may be expressed in terms of Fourier series as follows:

$y = A_0 + A_1 \sin \omega t + B_1 \cos \omega t + A_2 \sin 2\omega t + B_2 \cos 2\omega t + \cdots$

In the absence of external input (r=0), the output y of N is feedback to its input through the linear element G(S) tandem. It can be assumed to a good degree of accuracy that all the harmonics of y are filtered out in the process such that input x(t) to the non linear element N is mainly contributed by the fundamental component of y ,i.e.(t) remains sinusoidal. Under such conditions the harmonic content of y can be thrown away for the purpose of analysis and the fundamental component of y, i.e.,

$$y_1 = A_1 \sin \omega t + B_1 \cos \omega t$$

$$= Y_1 \sin(\omega t + \phi_1)$$

The non linearity can be replaced by a described function $K_N(X, \omega)$ which is defined to be the complex function embodying amplification and phase shift of the fundamental frequency component of y relative to x, i.e.

$$K_N(X, \omega) = \left(\frac{Y_1}{X}\right) \angle \phi_1$$

Derivation of Describing Functions

The describing function of a non linear element is given by

$$K_N(X, \omega) = \left(\frac{Y_1}{X}\right) \angle \phi_1$$

Where X =amplitude of the input sinusoid; Y1=amplitude of fundamental harmonic component of the output; and ϕ_1 =phase shift of fundamental harmonic component of the output with respect to input.

Therefore, for computing the describing function of a nonlinear element, we are simply required to find the fundamental harmonic component of its output for an input x= X $\sin \omega t$ .The fundamental component of the output can be written as

$$y_1 = A_1 \sin \omega t + B_1 \cos \omega t$$

Where the coefficients A_1 and B_2 of the Fourier series are

$$B_1 = \frac{1}{\pi} \int_0^{2\pi} y \cos \omega t \, d(\omega t) \quad \ldots\ldots\ldots\ldots (1)$$

$$A_1 = \frac{1}{\pi}\int_0^{2\pi} y \sin\omega t \, d(\omega t) \quad \ldots\ldots\ldots\ldots (2)$$

The amplitude and phase angle of the fundamental component of the output are given by

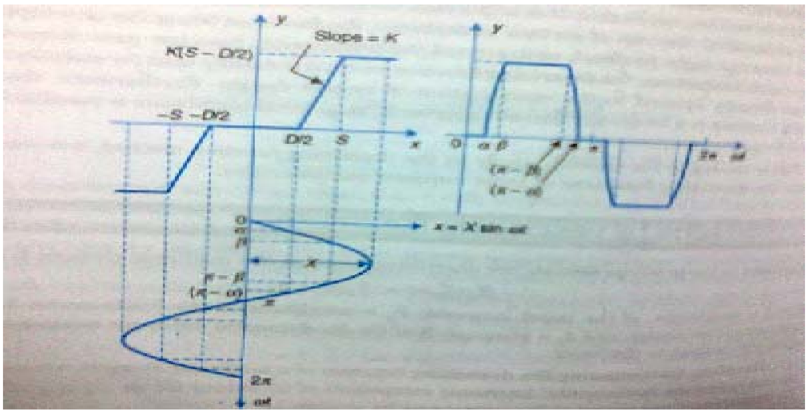
$$Y_1 = \sqrt{(A_1^2 + B_1^2)}$$

$$\phi_1 = \tan^{-1}\left(\frac{B_1}{A_1}\right)$$

Illustrative derivation of describing functions of commonly encountered non linearities are given below

Dead-zone and Saturation:

Idealized characteristics of a nonlinearity having dead-zone and saturation and its response to sinusoidal input are shown in figure. The output wave form may be described as follows:



$y = \{■(0 \qquad ,0 \le \omega t \le \alpha @K(x \quad D/2)$

Where $\alpha = \sin^{-1}(D/2X)$ and $\beta = \sin^{-1}(S/X)$

Using equations 1 and 2 recognizing that the output has half-wave and quarter-wave symmetries, we have

$B_1 = 0$

$$A_1 = \frac{4}{\pi}\int_0^{\pi/2} y \sin\omega t \, d(\omega t)$$

$$= \frac{4}{\pi}\int_\alpha^\beta K(X\sin\omega t - D/2)\sin\omega t \, d(\omega t) + \frac{4}{\pi}\int_\beta^{\pi/2} K(S - D/2)\sin\omega t \, d(\omega t)$$

$$= \frac{K}{\pi}\left\{2X(\beta-\alpha) - X(\sin 2\beta - \sin 2\alpha) + 4\left[\frac{D}{2}(\cos\beta - \cos\alpha) + \left(s - \frac{D}{2}\right)\cos\beta\right]\right\}$$

$$= \frac{KX}{\pi}[2(\beta-\alpha) + (\sin 2\beta - \sin 2\alpha)]$$

Therefore, the describing function is given by

$$\frac{K_N(X)}{K} = \begin{cases} 0 & ; X < (D/2); \alpha = \beta = \frac{\pi}{2} \\ 1 - \frac{2}{\pi}(\alpha + \sin\alpha\cos\alpha) & ; D/2 < X < S; \beta = \frac{\pi}{2} \\ \frac{1}{\pi}[2(\beta-\alpha) + (\sin 2\beta - \sin 2\alpha)] & ; X > S \end{cases} \quad \ldots\ldots (3)$$

Two special cases immediately from eqn……. (3)

Case-1 saturation nonlinearity $(D/2 = 0, \alpha = 0)$

$(K_1N(X))/K = \{■(1 \qquad ;X < S@2/\pi(\beta + \sin\beta \ \cos\beta)$

Case-2 Dead-zone nonlinearity $(S \to \infty, \beta - \pi/2)$

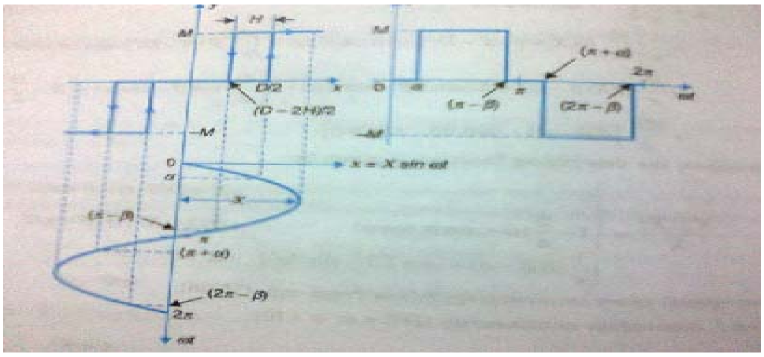
$(K_1N(X))/K = \{■(0 \qquad ;X < (D/2)@ \ 1 - 2/\pi(\alpha + \sin\alpha \ \cos\alpha)$

It is found that the describing functions of saturation and dead-zone nonlinearity are frequency-invariant having zero phase shifts. In fact all nonlinearities, whose input output characteristics are represented by a planner graph, would result in describing functions independent of frequency and amplitude dependent.

Lecture-9
Relay with Dead-zone and Hysteresis, Backlash

Relay with Dead-zone and Hysteresis:

The characteristics of a relay with dead-zone and hysteresis and its response to sinusoidal input are sown in figure.



The output y may be described as follows:

$$y = \begin{cases} 0 & ; 0 \le \omega t \le \alpha \\ +M & ; \alpha \le \omega t \le \pi - \beta \\ 0 & ; \pi - \beta \le \omega t \le (\pi + \alpha) \\ -M & ; (\pi - \alpha) \le \omega t \le (2\pi - \beta) \\ 0 & ; (2\pi - \beta) \le \omega t \le 2\pi \end{cases}$$

Where $\alpha = \sin^{-1}(D/2X)$ and $\beta = \sin^{-1}([(D - 2H)/2X])$

But we have

$$B_1 = \frac{1}{\pi}\int_0^{2\pi} y \cos \omega t \, d(\omega t)$$

$$B_1 = \frac{2}{\pi}\int_{\alpha}^{\pi-\beta} M \cos \omega t \, d(\omega t)$$

$$= \frac{2M}{\pi}(\sin\beta - \sin\alpha)$$

$$= \frac{2M}{\pi}\left(\frac{H}{X}\right)$$

$$A_1 = \frac{1}{\pi}\int_0^{2\pi} y \sin \omega t \, d(\omega t)$$

$$= \frac{2}{\pi}\int_{\alpha}^{\pi-\beta} M \sin\omega t \, d(\omega t)$$

$$= \frac{2M}{\pi}(\cos\alpha + \cos\beta)$$

$$= \frac{2M}{\pi}\left\{\sqrt{\left[1 - \left(D/_{2X}\right)^2\right]} + \sqrt{\left[1 - \left((D-2H)/_{2X}\right)^2\right]}\right\}$$

Therefore

$$\frac{B_1}{X} = \frac{2M}{\pi X}\left(-\frac{H}{X}\right) \cdots\cdots (4)$$

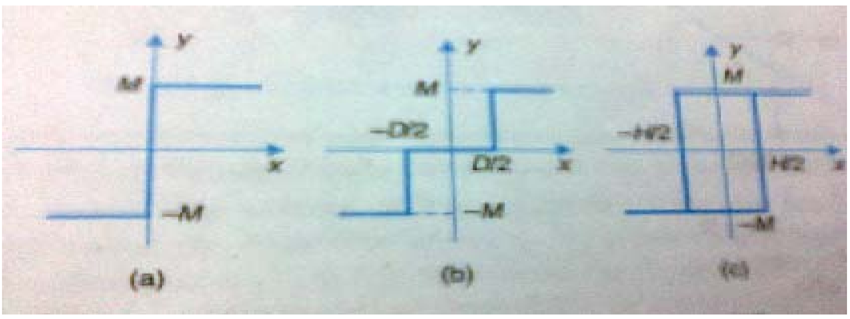
$$\frac{A_1}{X} = \frac{2M}{\pi X}\left\{\sqrt{\left[1 - \left(D/_{2X}\right)^2\right]} + \sqrt{\left[1 - \left((D-2H)/_{2X}\right)^2\right]}\right\} \cdots (5)$$

$$K_N(X) = \begin{cases} \sqrt{\left[\left(A_1/_X\right)^2 + \left(B_1/_X\right)^2\right]} \angle \tan^{-1}\left(\frac{B_1}{A_1}\right); X > D/_2 \\ 0 \qquad ; X < D/_2 \end{cases} \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

It is seen that K_N is a function of X, the input amplitude and is independent of frequency. Further, it being memory type nonlinearity, K_N has both magnitude and angle .Equation (6) is the general expression for the describing function of a relay element, from which the describing functions of various simplified relay characteristics given below;

1. Ideal relay,(fig a) letting D=H=0 in eqn.(6)

$$K_N(X) = \frac{4M}{\pi X} \ldots\ldots\ldots\ldots (7)$$



2. Relay with dead-zone (fig b).letting H=0 in eqn. (6)

$$\frac{D}{M}K_N(X) - \begin{cases} 0; X < \frac{D}{2} \\ \frac{4D}{\pi X}\sqrt{\left[1-\left(\frac{D}{2X}\right)^2\right]} \end{cases}$$; X>D/2

3. Relay with hysteresis (fig c).Letting H=D in eqn. (6)

$$\frac{H}{M}K_N(X) = \begin{cases} 0; X < \frac{D}{2} \\ \frac{4H}{\pi X} < -\sin^{-1} \frac{H}{2X} \end{cases}$$; X>H/2

Backlash

The characteristics of backlash nonlinearity and its response to sinusoidal input are shown in fig. The output may be described as follows:

y =x - b/2 ; $0 \le \omega t \le \pi/2$

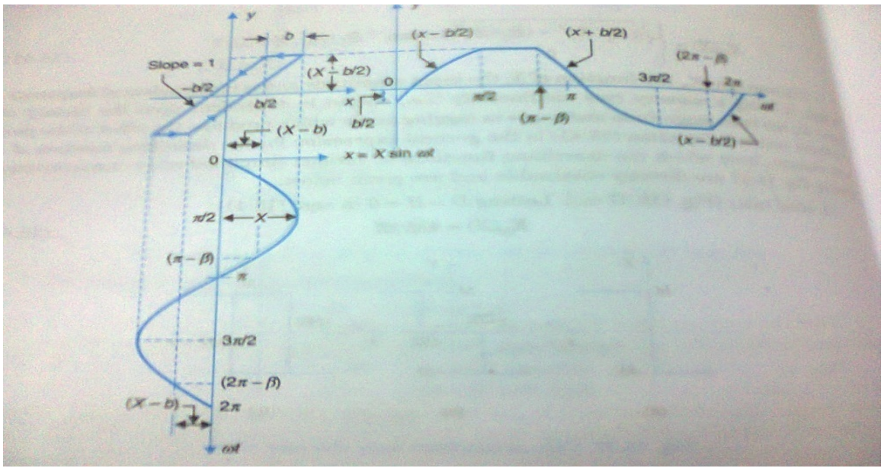
=X - b/2 ; $\pi/2 \le \omega t \le (\pi - \beta)$

=x + b/2 ; $(\pi - \beta) \le \omega t \le 3\pi/2$

=-X + b/2 ; $3\pi/2 \le \omega t \le (2\pi - \beta)$

=x - b/2 ; $(2\pi - \beta) \le \omega t \le 2\pi$

Where $\beta = \sin^{-1}\left(1 - \frac{b}{X}\right)$



$$B_1 = \frac{2}{\pi}\int_0^{\pi} y \cos \omega t \, d(\omega t)$$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \left(X \sin wt - \frac{b}{2}\right) \cos \omega t \, d(\omega t) + \frac{2}{\pi}\int_{\alpha}^{\pi-\beta} \left(X - \frac{b}{2}\right) \cos \omega t \, d(\omega t) +$$

$$\frac{2}{\pi}\int_{\pi-\beta}^{\alpha} \left(X \sin wt + \frac{b}{2}\right) \cos \omega t \, d(\omega t)$$

$$= -\frac{X}{\pi} \cos^2 \beta$$

$$A_1 = \frac{2}{\pi}\int_0^{\pi} y \sin \omega t \, d(\omega t)$$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \left(X \sin wt - \frac{b}{2}\right) \sin \omega t \, d(\omega t) + \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\beta} \left(X - \frac{b}{2}\right) \sin \omega t \, d(\omega t) +$$

$$\frac{2}{\pi}\int_{\pi-\beta}^{\pi} \left(X \sin wt - \frac{b}{2}\right) \sin \omega t \, d(\omega t)$$

$$= \frac{X}{\pi}\left[\left(\frac{\pi}{2} + \beta\right) + \frac{1}{2}\sin 2\beta\right]$$

Therefore,

$$K_N(X) = \begin{cases} \sqrt{\left[\left(A_1/_X\right)^2 + \left(B_1/_X\right)^2\right]} < \tan^{-1}\left(\frac{B_1}{A_1}\right); X > b/_2 \\ 0 \qquad ; X < b/_2 \end{cases}$$

Lecture-10
Stability Analysis by Describing Function Method, Jump Resonance

Describing functions are most frequently used to determine if limit cycles (stable-amplitude periodic oscillations) are possible for a given system, and to determine the amplitudes of various signals when these oscillations are present.

Describing-function analysis is simplified if the system can be arranged in a form similar to that shown in Fig. **6.9.** The inverting block is included to represent the inversion conventionally indicated at the summing point in a negative-feedback system. Since the intent of the analysis is to examine the possibility of steady-state oscillations, system input and output points are irrevelant. The important feature of the topology shown in Fig. **6.9** is

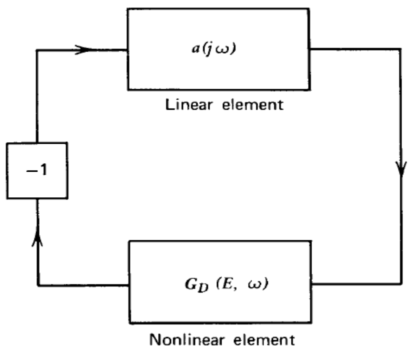
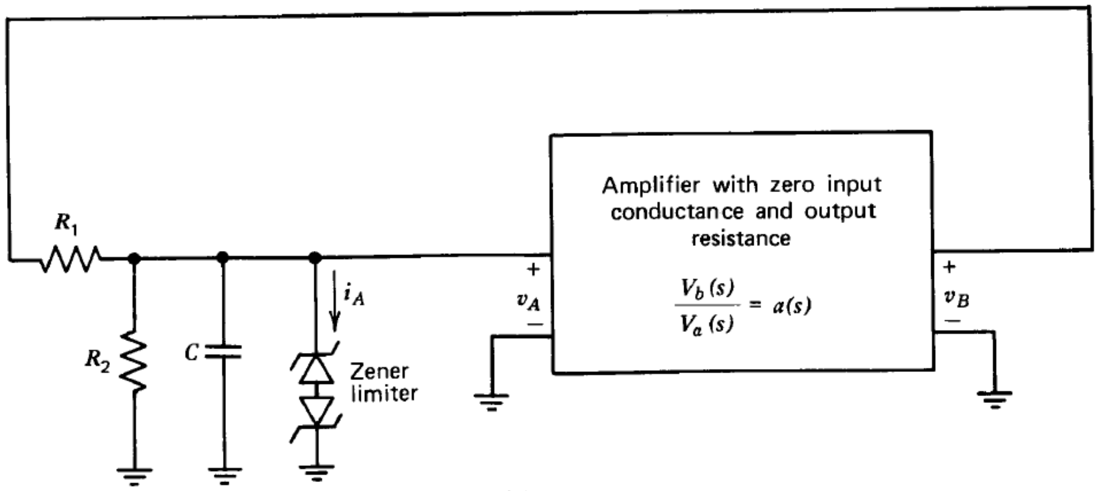


Figure 6.9 System arranged for describing-function analysis.

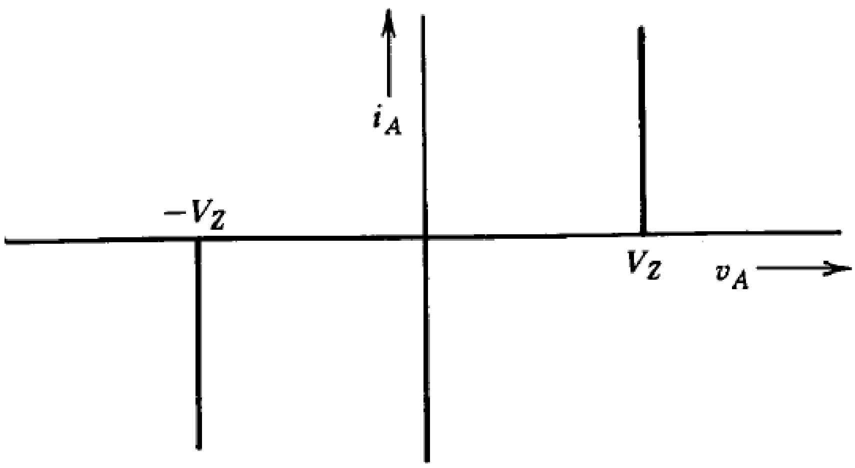
that a single nonlinear element appears in a loop with a single linear element. The linear element shown can of course represent the reduction of a complex interconnection of linear elements in the original system to a single transfer function.

The system shown in Fig. **6.10** illustrates a type of manipulation that simplifies the use of describing functions in certain cases. **A** limiter consisting of back-to-back Zener diodes is included in a circuit that also consisting of back-to-back Zener diodes is included in a circuit that also contains an amplifier and a resistor-capacitor network. The Zener limiter is assumed to have the piecewise-linear characteristics shown in Fig. **6.10b.**

The describing function for the nonlinear network that includes ***R1, R2, C,*** and the limiter could be calculated **by** assuming a sinusoidal signal for v_b and finding the amplitude and relative phase angle of the fundamental component **of** *VA*. The resulting describing function would be frequency.



(a)

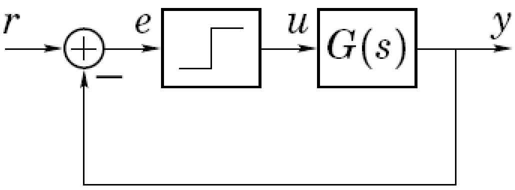


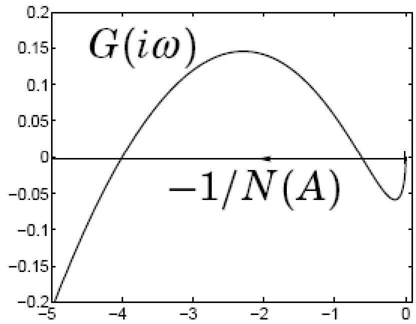
(b)

Once a system has been reduced to the form shown in Fig. **6.9,** it can be analyzed **by** means of describing functions. The describing-function approximation states that oscillations may be possible if particular values of $\mathbf{E_i}$ and ω_1 exist such that

$$a(j\omega_1)G_D(E_1,\ \omega_1) = -1$$

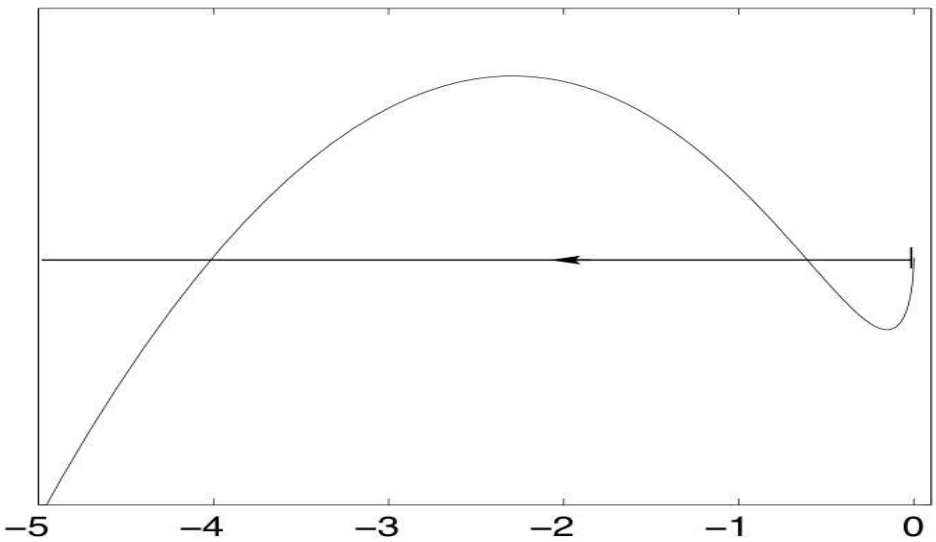
$$a(j\omega_1) = \frac{-1}{G_D(E_1,\ \omega_1)}$$





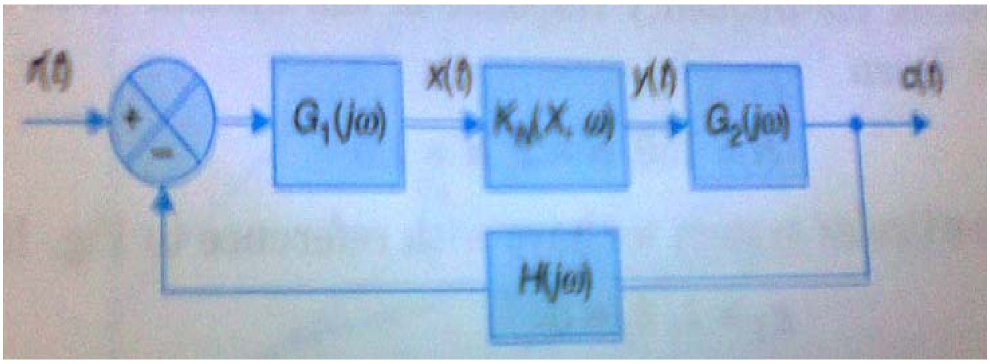
$$G(s) = \frac{(s+10)^2}{(s+1)^3} \quad \text{with feedback} \quad u = -\text{sgn}\, y$$

Gives one stable and one unstable limit cycle. The left most intersection corresponds to the stable one.



Jump Resonance

Consider a nonlinear system shown in figure where in the nonlinear part has been replaced by its describing function.



Let r (t) = R $\sin\omega t$

X (t) = R$X\sin(\omega t + \emptyset)$

It immediately follows that

$$\frac{C(j\omega)}{R(j\omega)} = \frac{G_1(j\omega)K_N(X,\omega)G_2(j\omega)}{1 + G_1(j\omega)K_N(X,\omega)G_2(j\omega)H(j\omega)}$$

We can also write

$$\frac{X(j\omega)}{R(j\omega)} = \frac{G_1(j\omega)}{1 + G_1(j\omega)K_N(X,\omega)G_2(j\omega)H(j\omega)}$$

Further we can write

$G_1(j\omega) = g_1(\omega)exp\{j\theta_1(\omega)\}$

$G_1(j\omega)G_2(j\omega)H(j\omega) = g_2(\omega)exp\{j\theta_2(\omega)\}$

$K_N(X,\omega) = g_N(X,\omega)exp\{j\theta_N(\omega)\}$

In order to simplify the analysis we assume

(i) $\theta_N(X,\omega) = 0$

(ii) $g_N(X,\omega) = g_N(X)$ i.e independent of frequency

So we can write

$$\frac{X(j\omega)}{R(j\omega)} = \frac{g_1(\omega)exp\{j\theta_1(\omega)\}}{1 + g_2(\omega)g_N(X)exp\{j\theta_2(\omega)\}}$$

In terms of magnitudes, we can write the above expression as

$$\frac{X}{R} = \frac{g_1}{\sqrt{1 + 2g_N g_2 \cos\theta_2 + g_N^2 g_2^2}}$$

Solving for g_N, we get

$$g_N = \frac{-\cos\theta_2 \pm \sqrt{\cos^2\theta_2 - 1 + \left(\frac{g_1^2 R^2}{X^2}\right)}}{g_2}$$

L ecture-11
Basic Stability Theores, Liapunov Functions, Instability

Consider a dynamical system which satisfies

$$\dot{x} = f(x, t) \qquad x(t_0) = x_0 \qquad x \in R_n. \ (4.31)$$

We will assume that $f(x, t)$ satisfies the standard conditions for the existence and uniqueness of solutions. Such conditions are, for instance, that $f(x, t)$ is Lipschitz continuous with respect

to x, uniformly in t, and piecewise continuous in t. A point $x^* \in Rn$ is an *equilibrium point* of

(4.31) if

$F(x^*, t) \equiv 0$. Intuitively and somewhat crudely speaking, we say an equilibrium point is *locally stable* if all solutions which start near x^* (meaning that the initial conditions are in a

neighborhood of $x*$) remain near $x*$ for all time. The equilibrium point $x*$ is said to be *locally*

asymptotically

stable if $x*$ is locally stable and, furthermore, all solutions starting near $x*$ tend towards $x*$ as

$t \rightarrow \infty$. We say somewhat crude because the time-varying nature of equation (4.31) introduces all kinds of additional subtleties. Nonetheless, it is intuitive that a pendulum has a locally stable
equilibrium point when the pendulum is hanging straight down and an unstable equilibrium point when it is pointing straight up. If the pendulum
is damped, the stable equilibrium point is locally asymptotically stable.

By shifting the origin of the system, we may assume that the equilibrium

point of interest occurs at $x* = 0$. If multiple equilibrium points exist, we will need to study

the stability of each by appropriately shifting the origin.

4.1 Stability in the sense of Lyapunov

The equilibrium point $x* = 0$ of (4.31) is *stable (in the sense of Lyapunov)*

at $t = t_0$ if for any $\epsilon > 0$ there exists a $\delta(t_0, \epsilon) > 0$ such that

$$\|x(t_0)\| < \delta \quad \Rightarrow \quad \|x(t)\| < \epsilon \ , \ \forall t \geq t_0 \ldots\ldots\ldots.(4.32)$$

Lyapunov stability is a very mild requirement on equilibrium points. In particular, it does not require that trajectories starting close to the origin tend to the origin asymptotically. Also, stability is defined at a time instant t_0. *Uniform stability* is a concept which guarantees that the equilibrium point is not losing stability. We insist that for a uniformly stable equilibrium

point $x*$, δ in the Definition 4.1 not be a function of t_0, so that equation (4.32) may hold for

all t_0. Asymptotic stability is made precise in the following definition:

4.2 Asymptotic stability

An equilibrium point $x^* = 0$ of (4.31) is *asymptotically stable* at $t = t_0$ if

1. $x^* = 0$ is stable, and
2. $x^* = 0$ is locally attractive; i.e., there exists $\delta(t_0)$ such that

$$\|x(t_0)\| < \delta \Rightarrow \lim_{t\to\infty} x(t) = 0 \qquad (4.33)$$

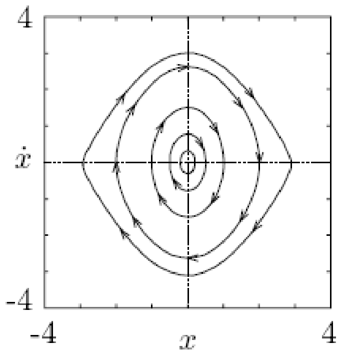
As in the previous definition, asymptotic stability is defined at t_0.*Uniform asymptotic stability* requires:

1. $x^* = 0$ is uniformly stable, and
2. $x^* = 0$ is uniformly locally attractive; i.e., there exists δ independent of t_0 for which equation (4.33) holds. Further, it is required that the convergence in equation (4.33) is uniform.

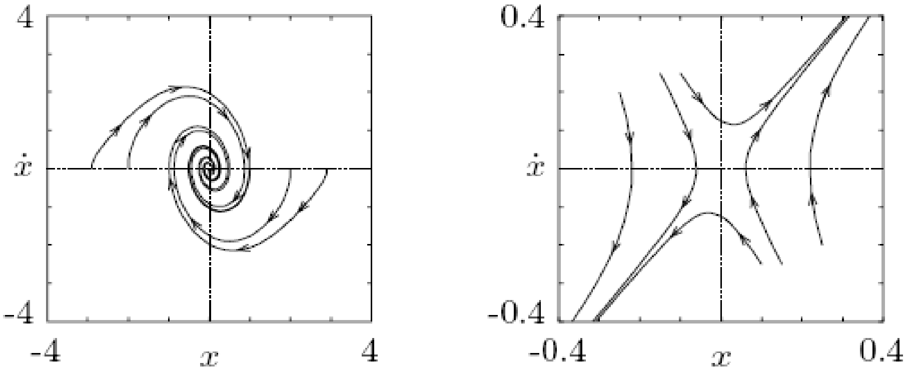
Finally, we say that an equilibrium point is *unstable* if it is not stable. This is less of a tautology than it sounds and the reader should be sure he or she can negate the definition of stability in the sense of Lyapunov to get a definition of instability. In robotics, we are almost always interested in uniformly asymptotically stable equilibria. If we wish to move the robot to a point, we would like to actually converge to that point, not merely remain nearby. Figure 4.7 illustrates the difference between stability in the sense of Lyapunov and asymptotic stability. Definitions 4.1 and 4.2 are *local* definitions; they describe the behavior of a system near an equilibrium point. We say an equilibrium point x^* is *globally* stable if it is stable for

all initial conditions $x_0 \in \mathbb{R}^n$. Global stability is very desirable, but in many applications it can

be difficult to achieve. We will concentrate on local stability theorems and indicate where it is possible to extend the results to the global case. Notions



(a) Stable in the sense of Lyapunov



(b) Asymptotically stable (c) Unstable (saddle)

Table 4.1: Summary of the basic theorem of Lyapunov.

	Conditions on $V(x,t)$	Conditions on $-\dot{V}(x,t)$	Conclusions
1	lpdf	≥ 0 locally	Stable
2	lpdf, decrescent	≥ 0 locally	Uniformly stable
3	lpdf, decrescent	lpdf	Uniformly asymptotically stable
4	pdf, decrescent	pdf	Globally uniformly asymptotically stable

In what follows, by $\dot{V}$ we will mean $\dot{V}|_{\dot{x}=f(x,t)}$.

Basic theorem of Lyapunov

Let V (x, t) be a non-negative function with derivative ˙V along the trajectories of the system.

1. If V (x, t) is locally positive definite and ˙V (x, t) ≤ 0 locally in x and for all t, then the origin of the system is locally stable (in the sense of Lyapunov).

2. If V (x, t) is locally positive definite and decrescent, and ˙V (x, t) ≤ 0 locally in x and for all t, then the origin of the system is uniformly locally stable (in the sense of Lyapunov).

3. If V (x, t) is locally positive definite and decrescent, and − ˙V (x, t) is locally positive definite, then the origin of the system is uniformly locally asymptotically stable.

4. If V (x, t) is positive definite and decrescent, and − ˙V (x, t) is positive definite, then the origin of the system is globally uniformly asymptotically stable.

The conditions in the theorem are summarized in Table 4.1.

Theorm-1

Consider the system

$\dot{x} = f(x); \quad f(0) = 0$

Suppose there exists a scalar function v(x) which for some real number $\epsilon > 0$, satisfies the following properties for all x in the region $\|x(t)\| \leq \epsilon$

(a) V(x)>0; $x \neq 0$ *that is v(x) is positive definite scalar function.*

(b) V (0) = 0

(c)V(x) has continuous partial derivatives with respect to all component of x

(d) $\frac{dV}{dt} \leq 0$ *(i.e dv/dt is negative semi definite scalar function)*

Then the system is stable at the origin.

Theorem-2

If the property of (d) of theorem-1 is replaced with (d) $\frac{dV}{dt} < 0$ *,* $x \neq 0$ *(i.e dv/dt is negative definite scalar function),then the system is asymptotically stable.*

It is intuitively obvious since continuous v function>0 except at x=0, satisfies the condition dv/dt <0, we except that x will eventually approach the origin .We shall avoid the rigorous of this theorem.

Theorem-3

If all the conditions of theorem-2 hold and in addition.

$V(x) \to \infty \quad as \quad \|x\| \to \infty$

Then the system is asymptotically stable in-the-large at the origin.

Instability

It may be noted that instability in a nonlinear system can be established by direct recourse to the instability theorem of the direct method .The basic instability theorem is presented below:

Theorem-4 Consider a system

$\dot{x} = f(x); \quad f(0) = 0$

Suppose there a exist a scalar function W(x) which, for real number $\epsilon > 0$, satisfies the following properties for all x in the region $\|x\| \leq \epsilon$;

(a) W(x)>0; $x \neq 0$

(b) W (0) = 0

(c)W(x) has continuous partial derivatives with respect to all component of x

(d) $\frac{dW}{dt} \geq 0$

Then the system is unstable at the origin.

Lecture-12
Direct Method of Liapunov & the Linear System, Methods of constructing Liapunov functions for Non linear Systems.

Direct Method of Liapunov & the Linear System:

In case of linear systems, the direct method of liapunov provides a simple approach to stability analysis. It must be emphasized that compared to the results presented, no new results are obtained by the use of direct method for the stability analysis of linear systems. However, the study of linear systems using the direct method is quite useful because it extends our thinking to nonlinear systems.

Consider a linear autonomous system described by the state equation

$$\dot{x} = Ax \quad \text{...............................13.9}$$

The linear system is asymptotically stable in-the-large at the origin if and only if given any symmetric, positive definite matrix **Q,** there exists a symmetric positive definite matrix **P** which is the unique solution

$$A^T P + PA = -Q \qquad \ldots\ldots\ldots.13.10$$

Proof

To prove the sufficiency of the result of above theorem, let us assume that a symmetric positive definite matrix **P** exists which is the unique solution of eqn.(13.11). Consider the scalar function.

$$V(\mathbf{x}) = \mathbf{x}^T \mathbf{Px}$$

Note that $V(\mathbf{x}) > 0$ for $\mathbf{x} \neq \mathbf{0}$

$$V(\mathbf{0}) = \mathbf{0}$$

And

The time derivate of V(x) is

$$\dot{V}(\mathbf{x}) = \dot{x}^T \mathbf{P}_X + \mathbf{x}^T \mathbf{P}\dot{x}$$

Using eqns. (13.9) and (13.10) we get

$$V(\mathbf{x}) = \mathbf{x}^T \mathbf{A}^T \mathbf{Px} + \mathbf{x}^T \mathbf{PAx}$$

$$= \mathbf{x}^T(\mathbf{A}^T\mathbf{P} + \mathbf{PA})\mathbf{x}$$

$$= -\mathbf{x}^T\mathbf{Qx}$$

Since **Q** is positive definite, V(x) is negative definite. Norm of x may be defined as

Then $\|X\| = (x^T P_X)^{\frac{1}{2}}$

$$V(X) = \|X\|^2$$

$$V(X) \to \infty \ as \ \|X\| \to \infty$$

The system is therefore asymptotically stable in-the large at the origin.

In order to show that the result is also necessary, suppose that the system is asymptotically stable and P is negative definite, consider the scalar function

Therefore $V(\mathbf{x}) = \mathbf{x}^T \mathbf{Px} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.13.11$

$$\dot{V}(\mathbf{x}) = -[\dot{x}^T \mathbf{P}_X + \mathbf{x}^T \mathbf{P}\dot{x}]$$

$$= X^T Q_X$$

>0

There is contradiction since V(x) given by eqn. (13.11) satisfies instability theorem.

Thus the conditions for the positive definiteness of P are necessary and sufficient for asymptotic stability of the system of eqn. (13.9).

Methods of constructing Liapunov functions for Non linear Systems.

As has been said earlier ,the liapunov theorems give only sufficient conditions on system stability and furthermore there is no unique way of constructing a liapunov function except in the case of linear systems where a liapunov function can always be constructed and both necessary and sufficient conditions Established .Because of this draw back a host of methods have become available in literature and many refinements have been suggested to enlarge the region in which the system is found to be stable. since this treatise is meant as a first exposure of the student to the liapunov direct method, only two of the relatively simpler techniques of constructing a liapunov's function would be advanced here.

Krasovskiis method

Consider a system

$$\dot{x} = f(x); f(0) = 0$$

Define a liapunov function as

$$V = f^T P_f$$

Where P=a symmetric positive definite matrix.

Now

$$\dot{V} = \dot{f}^T P_f + f^T P \dot{f} \quad \text{.........13.13}$$

$$\dot{f} = \frac{\frac{df}{dx}dx}{dt} = Jf$$

$$J = \begin{bmatrix} \frac{df_1}{dx_1} & \frac{df_1}{dx_2} \cdots & \frac{df_1}{dx_n} \\ \frac{df_2}{dx_1} & \frac{df_2}{dx_2} \cdots & \frac{df_2}{dx_n} \\ \frac{df_n}{dx_1} & \frac{df_n}{dx_2} \cdots & \frac{df_n}{dx_n} \end{bmatrix} \text{ is jacobian matrix}$$

Substituting $\dot{f}$ in eqn (13.13), we have

$$\dot{V} = f^T J^T P f + f^T P J f$$

$= f^T (J^T P + PJ)$ f

Let Q $= J^T$ P + PJ

Since V is positive definite, for the system to be asymptotically stable, Q should be negative definite. If in addition, $V(x) \to \infty$ as $\|X\| \to \infty$, the system is asymptotically stable in-the-large.