BIJU PATNAIK UNIVERSITY OF TECHNOLOGY, ODISHA

Lecture Notes

On

THEORY OF COMPUTATION

MODULE -1

UNIT - 3

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UNIT 3 CONTEXT FREE GRAMMAR

Structure **Page Nos.**

3.0 INTRODUCTION

In unit 2, we studied the class of regular languages and their representations through regular expressions and finite automata. We have also seen that not all languages are regular. If a language is not regular than there should be other categories of language also. We have also seen that languages are defined by regular expression. Regular languages are closed under union, product, Kleene star, intersection and complement. Application areas are: text editors, sequential circuits, etc. The corresponding acceptor is Finite Automata.

Now, we shall discuss the concept of context free grammar for a larger class of languages. Language will be defined by context free grammar. Corresponding acceptor is Pushdown Automata. In this unit we shall check whether a context free language is closed under union, product and Kleene star or not. Language that will be defined by context free grammar is context free language. Application areas are: programming languages, statements and compilers.

3.1 OBJECTIVES

After studying this unit, you should be able to

- create a grammar from language and vice versa;
- explain and create context free grammar and language;
- define the pushdown automata;
- apply the pumping lemma for non-context free languages; and
- find the equivalence of context free grammar and Pushdown Automata

In unit 1, we discussed language and a regular language. A language in meaning if a grammar is used to derive the language. So, it is very important to construct a language from a grammar. As you know all languages are not regular. This non-regular languages are further categorised on the basis of classification of grammar.

3.2 GRAMMAR AND ITS CLASSIFICATION

In our day-to-day life, we often use the common words such as grammar and language. Let us discuss it through one example.

Example 1: If we talk about a sentence in English language, “Ram reads”, this sentence is made up of Ram and reads. Ram and reads are replaced for <noun> and <verb>. We can say simply that a sentence is changed by noun and verb and is written as

<sentence> → <noun> <verb>

where noun can be replaced with many such values as Ram, Sam, Gita…. and also <verb> can be replaced with many other values such as read, write, go …. As noun and verb are replaced, we easily write

<noun>	→	I
<noun>	→	Ram
<noun>	→	Sam
<verb>	→	reads
<verb>	→	writes

From the above, we can collect all the values in two categories. One is with the parameter changing its values further, and another is with termination. These collections are called variables and terminals, respectively. In the above discussion variables are, <sentence>, <noun> and <verb>, and terminals are I, Ram, Sam, read, write. As the sentence formation is started with <sentence>, this symbol is special symbol and is called **start symbol**.

Now **formally**, a Grammar G = (V, Σ, P, S) where,

- V is called the set of variables. e.g., {S, A, B, C}
- Σ is the set of terminals, e.g. {a, b}
- P is a set of production rules
  (- Rules of the form $A \rightarrow \alpha$ where $A \in (VU\Sigma)^+$ and $\alpha \in (VU\Sigma)^+$ e.g., $S \rightarrow aA$).
- S is a special variable called the start symbol $S \in V$.

Structure of grammar: If L is a language over an alphabet A, then a grammar for L consists of a set of grammar rules of the form

$$x \rightarrow y$$

where x and y denote strings of symbols taken from A and from a set of grammar symbols disjoint from A. The grammar rule $x \rightarrow y$ is called a production rule, and application of production rule (x is replaced by y), is called derivation.

Every grammar has a special grammar symbol called the start symbol and there must be at least one production with the left side consisting of only the start symbol. For example, if S is the start symbol for a grammar, then there must be at least one production of the form $S \rightarrow y$.

Example 2: Suppose A = {a, b, c} then a grammar for the language A^* can be described by the following four productions:

$S \rightarrow$	$\wedge$	(i)
$S \rightarrow$	aS	(ii)
$S \rightarrow$	bS	(iii)
$S \rightarrow$	cS	(iv)

$$S \underset{\text{using prod.(u)}}{\Rightarrow} aS \underset{\text{using prod.(ii)}}{\Rightarrow} aaS \underset{\text{using prod.(iv)}}{\Rightarrow} aacS \underset{\text{using prod.(iii)}}{\Rightarrow} aacbS \underset{\text{using prod.(i)}}{\Rightarrow} aacb = aacb$$

The desired derivation of the string is aacb. Each step in a derivation corresponds to a branch of a tree and this true is called **parse tree**, whose root is the start symbol. The completed derivation and parse tree are shown in the Figure 1,2,3:

Fig. 1: S ⇒ aS

Fig. 2: S ⇒ aS ⇒ aaS

Fig. 3: S ⇒ aS ⇒ aaS ⇒ aacS

Let us derive the string aacb, its parse tree is shown in figure 4.

$S \Rightarrow aS \Rightarrow aaS \Rightarrow aacS \Rightarrow aacbS \Rightarrow aacb\wedge = aacb$

Fig. 4: Parse tree deriving aacb

Sentential Form: A string made up of terminals and/or non-terminals is called a sentential form.

In example 1, formally grammar is rewritten as
In G = (V, Σ, P, S) where
V = {<sentence>, <noun>, <verb>}
Σ = {Ram, reads,…}
P = <sentence> → <noun> <verb>
<noun> → Ram
<verb> → reads, and
S = <sentence>
If x and y are sentential forms and $\alpha \rightarrow \beta$ is a production, then the replacement of α by β in $x\alpha y$ is called a derivation, and we denote it by writing

$$x\alpha y \Rightarrow x\beta y$$

To the left hand side of the above production rule x is left context and y is right context. If the derivation is applied to left most variable of the right hand side of any

production rule, then it is called leftmost derivation. And if applied to rightmost then is called rightmost derivation.

The language of a Grammar :

A language is generated from a grammar. If G is a grammar with start symbol S and set of terminals Σ, then the language of G is the set

$$L(G) = \{W \mid W \in \Sigma^* \text{ and } S \underset{G}{\overset{*}{\Rightarrow}} W\}.$$

Any derivation involves the application production Rules. If the production rule is applied once, then we write $\alpha \underset{G}{\Rightarrow} B$. When it is more than one, it is written as $\alpha \underset{a}{\overset{*}{\Rightarrow}} \beta$

Recursive productions: A production is called recursive if its left side occurs on its right side. For example, the production $S \rightarrow aS$ is recursive. A production $A \rightarrow \alpha$ is indirectly recursive. If A derives a sentential form that contains A, Then, suppose we have the following grammar:

$$S \rightarrow b/aA$$
$$A \rightarrow c/bS$$

the productions $S \rightarrow aA$ and $A \rightarrow bs$ are both indirectly recursive because of the following derivations:

$$S \Rightarrow aA \Rightarrow abS,$$
$$A \Rightarrow bS \Rightarrow baA$$

A grammar is recursive if it contains either a recursive production or an indirectly recursive production.

A grammar for an infinite language must be recursive.

Example 3: Consider $\{\wedge, a, aa, \ldots, a^n, \ldots\} = \{a^n \mid n \geq 0\}$.

Notice that any string in this language is either $\wedge$ or of the form ax for some string x in the language. The following grammar will derive any of these strings:

$$S \rightarrow \wedge/aS.$$

Now, we shall derive the string aaa:

$$S \Rightarrow aS \Rightarrow aaS \Rightarrow aaaS \Rightarrow aaa.$$

Example 4: Consider $\{\wedge, ab, aabb, \ldots, a^n b^n, \ldots\} = \{a^n b^n \mid n \geq 0\}$.

Notice that any string in this language is either $\wedge$ or of the form axb for some string x in the language. The following grammar will derive any of the strings:

$$S \rightarrow \wedge/aSb.$$

For example, we will derive the string aaabbb;

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aaaSbbb \Rightarrow aaabbb.$$

Example 5: Consider a language $\{\wedge, ab, abab, \ldots, (ab)^n, \ldots\} = \{(ab)^n \mid n \geq 0\}$.

Notice that any string in this language is either $\wedge$ or of the form abx for some string x in the language. The following grammar will derive any of these strings:

$$S \rightarrow \wedge/abS.$$

For example, we shall derive the string ababab:

$$S \Rightarrow abS \Rightarrow ababS \Rightarrow abababS \Rightarrow ababab.$$

Sometimes, a language can be written in terms of simpler languages, and a grammar can be constructed for the language in terms of the grammars for the simpler languages. We will now concentrate on operations of union, product and closure.

Suppose M and N are languages whose grammars have disjoint sets of non-terminals. Suppose also that the start symbols for the grammars of M and N are A and B, respectively. Then, we use the following rules to find the new grammars generated from M and N:

Union Rule: The language $M \cup N$ starts with the two productions

$$S \rightarrow A/B.$$

Product Rule: The language MN starts with the production.

$$S \rightarrow AB$$

Closure Rule: The language M^* starts with the production

$$S \rightarrow AS/\wedge.$$

Example 6: Using the Union Rule:

Let's write a grammar for the following language:

$$L = \{\wedge, a, b, aa, bb, \ldots, a^n, b^n, \ldots\}.$$

L can be written as union.

$$L = M \cup N,$$

Where $M = \{a^n \mid n \geq 0\}$ and $N = \{b^n \mid n \geq 0\}$.

Thus, we can write the following grammar for L:

$S \rightarrow A \mid B$ union rule,
$A \rightarrow \wedge/aA$ grammar for M,
$B \rightarrow \wedge/bB$ grammar for N.

Example 7: Using the Product Rule:

We shall write a grammar for the following language :

$$L = \{a^m b^n \mid m,n \geq 0\}.$$

L can be written as a product L = MN, where $M = \{a^m \mid m \geq 0\}$ and $N = \{b^n \mid n \geq 0\}$.
Thus we can write the following grammar for L:

$S \rightarrow AB$ product rule
$A \rightarrow \wedge/aA$ grammar for M,
$B \rightarrow \wedge/bB$ grammar for N,

Example 8: Using the Closure Rule: For the language L of all strings with zero or more occurrence of aa or bb. $L = \{aa, bb\}^*$. If we let $M = \{aa, bb\}$, then $L = M^*$.
Thus, we can write the following grammar for L:

$S \rightarrow AS/\wedge$ closure rule,
$A \rightarrow aa/bb$ grammar for M.

We can simplify the grammar by substituting for A to obtain the following grammar:

$$S \rightarrow aaS/bbS/\wedge$$

Example 9: Let $\Sigma = \{a, b, c\}$. Let S be the start symbol. Then, the language of palindromes over the alphabet Σ has the grammar.

$$S \rightarrow aSa/bSb/cSc/a/b/c/\wedge.$$

For example, the palindrome abcba can be derived as follows:

$$S \Rightarrow aSa \Rightarrow abSba \Rightarrow abcba$$

Ambiguity: A grammar is said to be ambiguous if its language contains some string that has two different parse tree. This is equivalent to saying that some string has two distinct leftmost derivations or that some string has two distinct rightmost derivations.

Example 10: Suppose we define a set of arithmetic expressions by the grammar:

$$E \rightarrow a/b/E–E$$

Fig. 5: Parse Tree **Fig. 6: Parse Tree showing ambiguity**

This is the parse tree for an ambiguous string.

The language of the grammar $E \rightarrow a/b/E\text{-}E$ contains strings like a, b, b–a, a–b–a, and b–b–a–b. This grammar is ambiguous because it has a string, namely, a–b–a, that has two distinct parse trees.

Since having two distinct parse trees mean the same as having two distinct left most derivations.

$$E \Rightarrow E–E \Rightarrow a–E \Rightarrow a–E–E \Rightarrow a – b – E \Rightarrow a – b – a.$$

$$E \Rightarrow E–E \Rightarrow E–E–E \Rightarrow a–E–E \Rightarrow a – b – E \Rightarrow a – b – a.$$

The same is the case with rightmost derivation.

- A derivation is called a leftmost derivation if at each step the leftmost non-terminal of the sentential form is reduced by some production.

- A derivation is called a rightmost derivation if at each step the rightmost non-terminal of the sentential form is reduced by some production.

Let us try some exercises.

Ex.1) Given the following grammar

$$S \rightarrow S[S]/\wedge$$

For each of the following strings, construct a leftmost derivation, a rightmost derivation and a parse tree.

(a) [ ] (b) [[ ]] (c) [ ] [ ] (d) [[] [[]]]

Ex.2) Find a grammar for each language

(a) $\{a^m b^n \mid m,n \in N, n>m\}$.
(b) $\{a^m bc^n \mid n \in N\}$.

Ex.3) Find a grammar for each language:

(a) The even palindromes over {a, b}.
(b) The odd palindromes over {a, b}.

Chomsky Classification for Grammar:

As you have seen earlier, there may be many kinds of production rules. So, on the basis of production rules we can classify a grammar. According to Chomsky classification, grammar is classified into the following types:

Type 0: This grammar is also called **unrestricted grammar**. As its name suggests, it is the grammar whose production rules are unrestricted.

All grammars are of type 0.

Type 1: This grammar is also called **context sensitive grammar**. A production of the form $xAy \rightarrow x\alpha y$ is called a type 1 production if $\alpha \neq \wedge$, which means length of the working string does not decrease.

In other words, $|xAy| \leq |x\alpha y|$ as $\alpha \neq \wedge$. Here, x is left context and y is right context.

A grammar is called type 1 grammar, if all of its productions are of type 1. For this, grammar $S \rightarrow \wedge$ is also allowed.

The language generated by a type 1 grammar is called a type 1 or **context sensitive language**.

Type 2: The grammar is also known as **context free grammar**. A grammar is called type 2 grammar if all the production rules are of type 2. A production is said to be of type 2 if it is of the form $A \rightarrow \alpha$ where $A \in V$ and $\alpha \in (V \cup \Sigma)^*$. In other words, the left hand side of production rule has no left and right context. The language generated by a type 2 grammar is called **context free language**.

Type 3: A grammar is called type 3 grammar if all of its production rules are of type 3. (A production rule is of type 3 if it is of form $A \rightarrow \wedge$, $A \rightarrow a$ or $A \rightarrow aB$ where $a \in \Sigma$ and $A,B \in V$), i.e., if a variable derives a terminal or a terminal with one variable. This type 3 grammar is also called **regular grammar**. The language generated by this grammar is called **regular language**.

Ex.4) Find the highest type number that can be applied to the following grammar:

(a) $S \rightarrow ASB/b,\ A \rightarrow aA$
(b) $S \rightarrow aSa/bSb/a/b/\wedge$
(c) $S \rightarrow Aa,\ A \rightarrow S/Ba,\ B \rightarrow abc.$

3.3 CONTEXT FREE GRAMMAR

We know that there are non-regular languages. For example: $\{a^n b^n \mid n \geq 0\}$ is non-regular language. Therefore, we can't describe the language by any of the four representations of regular languages, regular expressions, DFAs, NFAs, and regular grammars.

Language $\{a^n b^n \mid n \geq 0\}$ can be easily described by the non-regular grammar:

$$S \rightarrow \wedge / aSb.$$

So, a context-free grammar is a grammar whose productions are of the form :

$$S \rightarrow x$$

Where S is a non-terminal and x is any string over the alphabet of terminals and non-terminals. Any regular grammar is context-free. A language is context-free language if it is generated by a context-free grammar.

A grammar that is not context-free must contain a production whose left side is a string of two or more symbols. For example, the production $Sc \rightarrow x$ is not part of any context-free grammar.

Most programming languages are context-free. For example, a grammar for some typical statements in an imperative language might look like the following, where the words in bold face are considered to be the single terminals:

$$S \rightarrow \text{while E do S/ if E then S else S/\{SL\}/I: = E}$$

$$L \rightarrow SL/\wedge$$

$$E \rightarrow \ldots.\text{(description of an expression)}$$
$$I \rightarrow \ldots.\text{(description of an identifier).}$$

We can combine context-free languages by union, language product, and closure to form new context-free languages.

Definition: A context-free grammar, called a CFG, consists of three components:

1. An alphabet Σ of letters called terminals from which we are going to make strings that will be the words of a language.

2. A set of symbols called non-terminals, one of which is the symbols, start symbol.

3. A finite set of productions of the form

   One non-terminal $\rightarrow$ finite string of terminals and/or non-terminals.

Where the strings of terminals and non-terminals can consist of only terminals or of only non-terminals, or any combination of terminals and non-terminals or even the empty string.

The language generated by a CFG is the set of all strings of terminals that can be produced from the start symbol S using the productions as substitutions. A language generated by a CFG is called a context-free language.

Example 11: Find a grammar for the language of decimal numerals by observing that a decimal numeral is either a digit or a digit followed by a decimal numeral.

$$S \rightarrow D/DS$$
$$D \rightarrow 0/1/2/3/4/5/6/7/8/9$$

$$S \Rightarrow DS \Rightarrow 7S \Rightarrow 7DS \Rightarrow 7DDS \Rightarrow 78DS \Rightarrow 780S \Rightarrow 780D \Rightarrow 780.$$

Example 12: Let the set of alphabet A = {a, b, c}

Then, the language of palindromes over the alphabet A has the grammar:

$$S \rightarrow aSa \mid bSb \mid cSc \mid a \mid b \mid c \mid \wedge$$

For example, the palindrome abcba can be derived as follows:

$$P \Rightarrow aPa \Rightarrow abPba \Rightarrow abcba$$

Example 13: Let the CFG is $S \rightarrow L \mid LA$

$$A \rightarrow LA \mid DA \mid \wedge$$
$$L \rightarrow a \mid b \mid \ldots \mid Z$$
$$D \rightarrow 0 \mid 1 \mid \ldots \mid 9$$

The language generated by the grammar has all the strings formed by a, b,c ….z, 0, 1,…..9.

We shall give a derivation of string a2b to show that it is an identifier.

$$S \Rightarrow LA \Rightarrow aA \Rightarrow aDA \Rightarrow a2A \Rightarrow a2LA \Rightarrow a2bA \Rightarrow a2b$$

Context-Free Language: Since the set of regular language is closed under all the operations of union, concatenation, Kleen star, intersection and complement. The set of context free languages is closed under union, concatenation, Kleen star only.

Union

Theorem 1: if L_1 and L_2 are context-free languages, then L_1UL_2 is a context-free language.

Proof: If L_1 and L_2 are context-free languages, then each of them has a context-free grammar; call the grammars G_1 and G_2. Our proof requires that the grammars have no non-terminals in common. So we shall subscript all of G_1's non-terminals with a 1 and subscript all of G_2's non-terminals with a 2. Now. we combine the two grammars into one grammar that will generate the union of the two languages. To do this, we add one new non-terminal, S, and two new productions.

$$S \rightarrow S_1 \mid S_2$$

S is the starting non-terminal for the new union grammar and can be replaced either by the starting non-terminal for G_1 or for G_2, thereby generating either a string from L_1 or from L_2. Since the non-terminals of the two original languages are completely different, and once we begin using one of the original grammars, we must complete the derivation using only the rules from that original grammar. Note that there is no need for the alphabets of the two languages to be the same.

Concatenation

Theorem 2: If L_1 and L_2 are context-free languages, then L_1L_2 is a context-free language.

Proof : This proof is similar to the last one. We first subscript all of the non-terminals of G_1 with a 1 and all the non-terminals of G_2 with a 2. Then, we add a new nonterminal, S, and one new rule to the combined grammar:

$S \rightarrow S_1S_2$

S is the starting non-terminal for the concatenation grammar and is replaced by the concatenation of the two original starting non-terminals.

Kleene Star

Theorem 3: If L is a context-free language, then L^* is a context-free language.

Proof : Subscript the non-terminals of the grammar for L with a 1. Then add a new starting nonterminal, S, and the rules

$S \rightarrow S_1S$
$\quad | \ \Lambda$

The rule $S \rightarrow S_1S$ is used once for each string of L that we want in the string of L^*, then the rule $S \rightarrow \Lambda$ is used to kill off the S.

Intersection

Now, we will show that the set of context-free languages is not closed under intersection. Think about the two languages $L_1 = \{a^nb^nc^m \mid n,m \geq 0\}$ and $L_2 = \{a^mb^nc^n \mid n,m \geq 0\}$. These are both context-free languages and we can give a grammar for each one:

G_1:

$S \rightarrow AB$
$A \rightarrow aAb$
$\quad | \ \Lambda$
$B \rightarrow cB$
$\quad | \ \Lambda$

G_2:
$S \rightarrow AB$
$A \rightarrow aA$
$\quad | \ \Lambda$
$B \rightarrow bBc$
$\quad | \ \Lambda$

The strings in L_1 contain the same number of a's as b's, while the strings in L_2 contain the same number of b's as c's. Strings that have to be both in L_1 and in L_2, i.e., strings in the intersection, must have the same numbers of a's as b's and the same number of b's as c's.

Thus, $L_1 \cap L_2 = \{a^nb^nc^n \mid n \geq 0\}$. Using Pumping lemma for context-free languages it can be proved easily that $\{a^n b^n c^n \mid \ n \geq 0\}$ is not context-free language. So, the class of context-free languages is **not closed** under intersection.

Although the set is not closed under intersection, there are cases in which the intersection of two context-free languages is context-free. Think about regular languages, for instance. All regular languages are context-free, and the intersection of two regular languages is regular. We have some other special cases in which an intersection of two context-free languages is context, free.

Suppose that L_1 and L_2 are context-free languages and that $L_1 \subseteq L_2$. Then $L_2 \cap L_1 = L_1$ which is a context-free language. An example is EQUAL $\cap \{a^n b^n\}$. Since strings in $\{a^n b^n\}$ always have the same number of a's as b's, the intersection of these two languages is the set $\{a^n b^n\}$, which is context-free.

Another special case is the intersection of a regular language with a non-regular context-free language. In this case, the intersection will always be context-free. An example is the intersection of $L_1 = a^+ b^+ a^+$, which is regular, with L_2 = PALINDROME. $L_1 \cap L_2 = \{a^n b^m a^n \mid m,n \geq 0\}$. This language is context-free.

Complement

The set of context-free languages is not closed under complement, although there are again cases in which the complement of a context-free language is context-free.

Theorem 4: The set of context-free languages is not closed under complement.

Proof: Suppose the set is closed under complement. Then, if L_1 and L_2 are context-free, so are L_1' and L_2'. Since the set is closed under union, $L_1' \cup L_2'$ is also context-free, as is $(L_1' \cup L_2')'$. But, this last expression is equivalent to$L_1 \cap L_2$ which is not guaranteed to be context-free. So, our assumption must be incorrect and the set is not closed under complement.

Here is an example of a context-free language whose complement is not context-free. The language $\{a^n b^n c^n \mid n \geq 1\}$ is not context-free, but the author proves that the complement of this language is the union of seven different context-free languages and is thus context-free. Strings that are not in $\{a^n b^n c^n \mid n \geq 1\}$ must be in one of the following languages:

1. $M_{pq} = \{a^p b^q c^r \mid p,q,r \geq 1 \text{ and } p>q\}$ (more a's than b's)
2. $M_{qp} = \{a^p b^q c^r \mid p,q,r \geq 1 \text{ and } q>p\}$ (more b's than a's)
3. $M_{pr} = \{a^p b^q c^r \mid p,q,r \geq 1 \text{ and } s>r\}$ (more a's than c's)
4. $M_{rp} = \{a^p b^q c^r \mid p,q,r \geq 1 \text{ and } r>p\}$ (more c's than a's)
5. M = the complement of $a^+ b^+ c^+$ (letters out of order)

Using Closure Properties

Sometimes, we can use closure properties to prove that a language is not context-free. Consider the language our author calls DOUBLEWORD = $\{ww \mid w \in (a+b)^*\}$. Is this language context-free? Assume that it is. Form the intersection of DOUBLEWORD with the regular language $a^+ b^+ a^+ b^+$, we know that the intersection of a context-free language and a regular language is always context-free. The intersection of DOUBLEWORD and is $a^n b^m a^n b^m \mid n,m \geq 1\}$. But, this language is not context-free, so DOUBLEWORD cannot be context-free.

Think carefully when doing unions and intersections of languages if one is a superset of the other. The union of PALINDROME and $(a+b)^*$ is $(a+b)^*$, which is regular. So, sometimes the union of a context-free language and a regular language is regular. The union of PALINDROME and a^* is PALINDROME, which is context-free but not regular.

Now try some exercises:

Ex.5) Find CFG for the language over $\Sigma = \{a,b\}$.

(a) All words of the form

$a^x b^y a^z$, where x, y, z = 1,2,3… and y = 5x+7z

(b) For any two positive integers p and q, the language of all words of the form $a^x b^y a^z$, where x, y, z = 1, 2, 3… and y = px + qz.

3.4 PUSHDOWN AUTOMATA (PDA)

Informally, a pushdown automata is a finite automata with stack. The corresponding acceptor of context-free grammar is pushdown automata. There is one start state and there is a possibly empty-set of final states. We can imagine a pushdown automata as a machine with the ability to read the letters of an input string, perform stack operations, and make state changes.

The execution of a PDA always begins with one symbol on the stack. We should always specify the initial symbol on the stack. We assume that a PDA always begins execution with a particular symbol on the stack. A PDA will use three stack operations as follows:

(i) The **pop** operation reads the top symbol and removes it from the stack.
(ii) The **push** operation writes a designated symbol onto the top of the stack. For example, push (x) means put x on top of the stack.
(iii) The **nop** does nothing to the stack.

We can represent a pushdown automata as a finite directed graph in which each state (i.e., node) emits zero or more labelled edges. Each edge from state i to state j labelled with three items as shown in the Figure 7, where L is either a letter of an alphabet or $\wedge$, S is a stack symbol, and 0 is the stack operation to be performed.

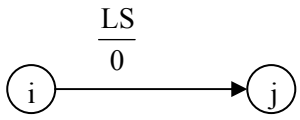


Fig. 7: Directed graph

It takes fine pieces of information to describe a labelled edge. We can also represent it by the following 5-tuple, which is called a PDA instruction.

(i, L, S, 0, j)

An instruction of this form is executed as follows, where w is an input string whose letters are scanned from left to right.

If the PDA is in state i, and either L is the current letter of w being scanned or L = $\wedge$, and the symbol on top of the stack is S, then perform the following actions:

(1) execute the stack operation 0;
(2) move to the state j; and
(3) if L $\neq \wedge$, then scan right to the next letter of w.

A string is accepted by a PDA if there is some path (i.e., sequence of instructions) from the start state to the final state that consumes all letters of the string. Otherwise, the string is rejected by the PDA. The language of a PDA is the set of strings that it accepts.

Nondeterminism: A PDA is deterministic if there is at most one move possible from each state. Otherwise, the PDA is non-deterministic. There are two types of non-determinism that may occur. One kind of non-determinism occurs exactly when a state emits two or more edges labelled with the same input symbol and the same stack symbol. In other words, there are two 5-tuples with the same first three components. For example, the following two 5-tuples represent nondeterminism:

(i, b, c, pop, j)

(i, b, c, push(D), k).

The second kind of nondeterminism occurs when a state emits two edges labelled with the same stack symbol, where one input symbol is $\wedge$ and the other input symbol is not. For example, the following two 5-tuples represent non-determinism because the machine has the option of consuming the input letter b or cleaning it alone.

(i, $\wedge$, c, pop, j)
(i, b, c, push(D), k).

Example 14: The language $\{a^n b^n \mid n \geq 0\}$ can be accepted by a PDA. We will keep track of the number of a's in an input string by pushing the symbol Y onto the stack for each a. A second state will be used to pop the stack for each b encountered. The following PDA will do the job, where x is the initial symbol on the stack:

Fig. 8: Pushdown Automata

The PDA can be represented by the following six instructions:

(0, $\wedge$, X, nop, 2)
(0, a, X, push(Y), 0),
(0, a, Y, push(Y), 0),
(0, b, Y, pop,1),
(1, b, Y, pop,1),
(1, $\wedge$, X, nop,2).

This PDA is non-deterministic because either of the first two instructions in the list can be executed if the first input letter is a and X is on the top of the stack. A computation sequence for the input string aabb can be written as follows:

(0, aabb, X) start in state 0 with X on the stack,
(0, abb, YX) consume a and push Y,
(0, bb, YYX) consume a and push Y,
(1, b, YX) consume b and pop.
(0, $\wedge$, X) consume b and pop .
(2, $\wedge$, X) move to the final state.

Equivalent Forms of Acceptance:

Above, we defined acceptance of a string by a PDA in terms of final state acceptance. That is a string is accepted if it has been consumed and the PDA is in a final state. But, there is an alternative definition of acceptance called empty stack acceptance, which requires the input string to be consumed and the stock to be empty, with no requirement that the machine be in any particular state. The class of languages accepted by PDAs that use empty stack acceptance is the same class of languages accepted by PDAs that use final state acceptance.

Example 15: (An empty stack PDA): Let's consider the language $\{a^n b^n \mid n \geq 0\}$, the PDA that follows will accept this language by empty stack, where X is the initial symbol on the stack.

Fig. 9: Pushdown Automata

PDA shown in figure 9 can also be represented by the following three instructions:

(0, a, X, push (X), 0),
(0, ∧, X, pop, 1),
(1, b, X, pop, 1).

This PDA is non-determinstic. Let's see how a computation proceeds. For example, a computation sequence for the input string aabb can be as follows:

(0, aabb, X) start in state 0 with X on the stack
(0, abb, XX) consume a and push X
(0, bb, XXX) consume a and push X
(1, bb, XX) pop.
(1, b, X) consume b and pop
(1, ∧, ∧) consume b and pop (stack is empty)

Now, try some exercises.

Ex.6) Build a PDA that accepts the language odd palindrome.

Ex.7) Build a PDA that accepts the language even palindrome.

3.5 NON-CONTEXT FREE LANGUAGES

Every context free grammar can always be represented in a very interesting form. This form is known as **Chomsky Normal Form (CNF)**.
A context-free grammar is said to be in Chomsky Normal Form if the right hand side of each production has either a terminal or two variables as $S \rightarrow a$, $S \rightarrow AB$ and $S \rightarrow \wedge$ if $\wedge \in L(G)$. If $\wedge \in L(G)$, then S should not appear to the right hand side of any production., To construct a CFG in, CNF we can develop a method. In CFG, $S \rightarrow a$ is already allowed, if the production is of form $S \rightarrow aA$ then can be replaced with $S \rightarrow BA$ and $B \rightarrow a$ in CNF. If the production is of the form $S \rightarrow ABC$, it can be written as $S \rightarrow AD$ and $D \rightarrow BC$. Using these simple methods, every CFG can be constructed in CNF.

Example 16: Reduce the following grammar, into CNF.

(i) $S \rightarrow a\,AB$, $A \rightarrow a\,E/bAE$, $E \rightarrow b$, $B \rightarrow d$
(ii) $S \rightarrow A0B$, $A \rightarrow AA/\ 0S/0$, $B \rightarrow 0BB/\ 1S/1$

Solution: (i) $S \rightarrow a\,AB$ is rewritten in CNF as $S \rightarrow FG$, $F \rightarrow a$ and $G \rightarrow AB$
$A \rightarrow aE$ is rewritten as $A \rightarrow FE$ in CNF.
$A \rightarrow bAE$ in CNF is $A \rightarrow HI$, $H \rightarrow b$ and $I \rightarrow AE$.
So Chomsky Normal Form of CFG is

$S \rightarrow FG$, $F \rightarrow a$, $G \rightarrow AB$, $A \rightarrow FE$, $A \rightarrow HI$,
$H \rightarrow b$, $I \rightarrow AE$, $E \rightarrow b$ and $B \rightarrow d$

(ii) left as an exercise.

In this section, we will prove that not all languages are context-free. Any context-free grammar can be put into Chomsky Normal Form. Here is our first theorem.

Theorem 5: Let G be a grammar in Chomsky Normal Form. Call the productions that have two non-terminals on the righthand side live productions and call the ones that have only a terminal on the right-hand side dead productions. If we are restricted to using the live productions of the grammar at most once each, we can generate only a finite number of words.

Proof: Each time when we use a live production, we increase the number of non-terminals in a working string by one. Each time when we use a dead production, we decrease the number of non-terminals by one. In a derivation starting with non-terminal S and ending with a string of terminals, we have to apply one more dead production than live production.

Suppose G has p live productions. Any derivation that does not reuse a live production can use at most p live and p+1 dead productions. Each letter in the final string results from one dead production, so words produced without reusing a live production must have no more than p+1 letters. There are a finite number of such words.

When doing a leftmost derivation, we replace the leftmost non-terminal at every step. If the grammar is in Chomsky Normal Form, each working string in a leftmost derivation is made up of a group of terminals followed by a group non-terminals. Such working strings are called *leftmost Chomsky working strings.*

Suppose we use a live production $Z \rightarrow XY$ twice in the derivation of some word w., Before the first use of $Z \rightarrow XY$ the working string has the form s_1Zs_2 where s_1 is a string of terminals and s_2 is a string of nonterminals. Before the second use of $Z \rightarrow XY$ the working string has form $s_1s_3Zs_4$ where s_3 is a string of terminals and s_4 is a string of non-terminals.

Suppose we draw a derivation tree representing the leftmost derivation in which we use $Z \rightarrow XY$ twice. The second Z we add to the tree could be a descendant of the first Z or it could come from some other nonterminal in s_2. Here are examples illustrating the two cases:

Case 1: Z is a descendant of itself.

$S \rightarrow AZ$
$Z \rightarrow BB$
$B \rightarrow ZA$
$\quad\quad \mid b$
$A \rightarrow a$

Beginning of a leftmost derivation:

$S \Rightarrow AZ$
$\Rightarrow aB$
$\Rightarrow aBB$
$\Rightarrow abB$
$\Rightarrow abZA$

Derivation tree is shown in figure 10:

Fig. 10: Leftmost derivation tree

Case 2: Z comes from a nonterminal in s_2.

$S \rightarrow AA$

$A \rightarrow ZC$

$\quad\quad | \ a$

$C \rightarrow ZZ$

$Z \rightarrow b$

Beginning of a leftmost derivation:

$S \Rightarrow AA$

$\Rightarrow ZCA$

$\Rightarrow bCA$

$\Rightarrow bZZA$

Derivation tree is shown in figure 11:

Fig. 11: Derivation Tree

In the first tree, Z is a descendant of itself. In the second, tree this is not true. Now, we will show that if a language is infinite, then we can always find an example of the first type of tree in the derivation tree of any string that is long enough.

Theorem 6: If G is a context-free grammar in Chomsky Normal Form that has p live productions, and if w is a word generated by G that has more than 2^p letters in it, then somewhere in every derivation tree for w there is an example of some non-terminal. Call it Z, being used twice where the second Z is descended from the first.

Proof: If the word w has more than 2^p letters in it, then the derivation tree for w has more than p+1 levels. This is because in a derivation tree drawn from a Chomsky Normal Form grammar, every internal node has either one or two children. It has one child only if that child is a leaf. At each level, there is at most twice the number of nodes as on the previous level. A leaf on the lowest level of the tree must have more than p ancestors. But, there are only p different live productions so if more than p have been used, then some live production has been used more than once. The non-terminal on the lefthand-side of this live production will appear at least twice on the path from the root to the leaf.

In a derivation, a non-terminal is said to be self-embedded if it ever occurs as a tree descendant of itself. The previous theorem says that in any context-free grammar, all sufficiently long words have leftmost derivations that include a self-embedded non-terminal. Shorter derivations may have self-embedded non-terminals, but we are guaranteed to find one in a sufficiently long derivation.

Consider the following example in which we find a self-embedded non-terminal:

$S \rightarrow AX$
$\quad | \; BY$
$\quad | \; AA$
$\quad | \; BB$
$\quad | \; a$
$\quad | \; b$
$X \rightarrow SA$
$Y \rightarrow SB$
$A \rightarrow a$
$B \rightarrow b$

Fig. 12: A derivation Tree for the String aabaa

The production $X \rightarrow SA$ and $S \rightarrow AX$ were used twice. Let's consider the X production and think about what would happens if we used this production a third time. What string would we generate? Corresponding tree is given in figure 13.

Fig. 13: Derivation Tree of String aaabaaa

This modified tree generates the string aaabaaa. We could continue reusing the rule $X \rightarrow SA$ over and over again. Can you tell what the pattern is in the strings that we would be producing?

The last use of X products the sub-string ba. The previous X produced an a to the left of this ba and an a to the right of the ba. The X before that produced an a to the left and an a to the right. In general, X produces $a^n baa^n$. S produces an a to the left of an X and nothing to the right. So, the strings produced by this grammar are of the form $aa^n baa^n$. If all we wish n to signify is a, count that must be the same, then we can

simplify this language description to a^nba^n for $n \geq 1$. Reusing the $X \rightarrow SA$ rule increases the number of a's in each group by one each time we use it.

Here is another example:

$S \rightarrow AB$
$A \rightarrow BC$
$\quad | \ a$
$B \rightarrow b$
$C \rightarrow AB$

In the derivation of the string bbabbb, $A \rightarrow BC$ is used twice. Look at the red triangular shapes in the following derivation tree. We could repeat that triangle more times and we would continue to generate words in the language.

Fig. 14: Derivation tree

Pumping Lemma for Context-Free Languages

Theorem 7: If G is any context-free grammar in Chomsky Normal Form with p live productions and w is any word generated by G with length $> 2^p$, we can subdivide w into five pieces uvxyz such that $x \neq \Lambda$, v and y are not both Λ and $|vxy| \leq 2^p$ and all words of the form uv^nxy^nz for $n \geq 0$ can also be generated by grammar G.

Proof: If the length of w is $> 2^p$, then there are always self-embedded non-terminals in any derivation tree for w. Choose one such self-embedded non-terminal, call it P, and let the first production used for P be $P \rightarrow QR$. Consider the part of the tree generated from the first P. This part of the tree tells us how to subdivide the string into its five parts. The sub-string vxy is made up of all the letters generated from the first occurrence of P. The sub-string x is made up of all the letters generated by the second occurrence of P. The string v contains letters generated from the first P, but to the left of the letters generated by the second P, and y contains letters generated by the first P to the right of those generated by the second P. The string u contains all letters to the left of v and the string z contains all letters to the right of y. By using the production $P \rightarrow QR$ more times, the strings v and y are repeated in place or "pumped". If we use the production $P \rightarrow QR$ only once instead of twice, the tree generates the string uxz.

Here is an example of a derivation that produces a self-embedded non-terminal and the resulting division of the string.

$S \rightarrow PQ$
$Q \rightarrow QS$
$\quad | \ b$
$p \rightarrow a$

Fig. 15: A derivation tree for the string abab

Notice that the string generated by the first occurrence of Q is bab. We have a choice for which Q we take for the second one. Let's first take the one to the far right. The string generated by this occurrence of Q is b. So x = b and v = ba. In this case, y is empty and so is z. The string u = a. If we pump v and y once, we get the string a | ba | ba | b = abababab which is also in the language. If we pump them three times, we get a | ba | ba | ba | b = abababab, etc.

Suppose we choose the other occurrence of Q for the second one, then we have a different sub-division of the string. In this case, the substring generated by the second occurrence of Q is b, so x = b and v is empty. The substring y, however, is ab in the case.

Fig. 16: Selection of u, x and y

If we pump v and y once, we get the string a | b | ab | ab = abababab; three times produces a | b | ab | ab | ab = abababababab, etc.

Using the Pumping Lemma for CFLs

We use the Pumping Lemma for context-free languages to prove that a language is **not** context-free. The proofs are always the same:

- Assume that the language in question is context-free and that the Pumping Lemma thus applies.
- Pick the string w, $|w| > 2^p$
- Sub-divide w into uvxyz such that $|vxy| < 2^p$

- Pick i so that uv^ixy^iz is not in the language. As in pumping lemma $uv^ixy^iz \in L$, but it is not true. So, our assumption is not correct and the language in the question is not CFL.

Here is an example:

Example 17: The language $L = \{a^nb^na^n \mid n \geq 1\}$ is not a context-free language.

Solution: Assume that L is a context-free language. Then, any string in L with length > 2^p can be sub-divided into uvxyz where uv^nxy^nz, n≥0, are all strings in the language. Consider the string $a^{2p}b^{2p}a^{2p}$ and how it might be sub-divided. Note that there is exactly one "ab" in a valid string and exactly one "ba". Neither v nor y can contain ab or ba or else pumping the string would produce more than one copy and the resulting string would be invalid. So both v and y must consist of all one kind of letters. There are three groups of letters all of which must have the same count for the string to be valid. Yet, there are only two sub-strings that get pumped, v and y. If we only pump two of the groups, we will get an invalid string.

A Stronger Version of the Pumping Lemma

There are times when a slightly stronger version of the Pumping Lemma is necessary for a particular proof. Here is the theorem:

Theorem 8: Let L be a context-free language in Chomsky Normal Form with p live productions. Then, any word w in L with length > 2^p can be sub-divided into five parts uvxyz such that the length of vxy is no more than 2^p, $x \neq \Lambda$, v and y are not both Λ, and uv^nxy^nz, n≥0, are all in the language L.

Now, let's see a proof in which this stronger version is necessary.

Example 18: The language $L = \{a^nb^ma^nb^m \mid n,m \geq 1\}$ is not context-free.

Proof: Assume that L is a context-free language. Then, any string in L with length > 2^p can be sub-divided into uvxyz where $x \neq \Lambda$, v and y are not both Λ, the length of vxy is no more than 2^p, and uv^nxy^nz, n≥0, are all strings in the language. Consider the string $a^{2p}b^{2p}a^{2p}b^{2p}$. (The superscripts on each character are supposed to be 2^p. Some browsers can't do the double superscript.) Clearly, this string is in L and is longer than 2^p. Since the length of vxy is no more than 2^p, there is no way that we can stretch vxy across more than two groups of letters. It is not possible to have v and y both made of a's, or v and y both made of b's. Thus, pumping v and y will produce strings with an invalid form. Note that we need the stronger version of the Pumping Lemma because without it we can find a way to sub-divide the string so that pumping it produces good strings. We could let u = Λ, v = the first group of a's, x = the first group of b's, y = the second group of a's, and z = the second group of b's. Now, duplicating v and y produces only good strings.

Here is another example.

Example 19: DOUBLEWORD = $\{ss \mid s \in \{a.b\}^*\}$ is not a context-free language.

Proof : The same proof as we used in the last case works here. Consider the string $a^{2p}b^{2p}a^{2p}b^{2p}$ (again supposed to be double superscripts.) It is not possible to have v and y both made of the same kind of letter, so pumping will produce strings that are not in DOUBLEWORD.

Now try some exercises

Ex.8) Show that the language $\{a^{n^2} \mid n \geq 1\}$ is not context free.

Ex.9) Show that the language $\{a^p \mid p \text{ is prime}\}$ is not context free.

3.6 EQUIVALENCE OF CFG AND PDA.

In Unit 2, we established the equivalence of regular languages/expressions and finite automata. Similarly, the context–free grammar and pushdown automata, models are equivalent in the sense these define the same set of languages.

Theorem 9: Every context free Grammar is accepted by some pushdown automata.

Let G = (V, T, R, S) be the given grammar
where the components have the following meaning

V : The set of variables
T : The set of terminals
R : The set of rules of the form
$A \to \alpha$ with $A \in V$ and $\alpha \in (V \cup T)^*$,
i.e., α is a string of terminals and non-terminals.
S : The start symbol.

Now, in terms of the given Grammer G, we achieve our goal through the following three parts:

(i) We construct a PDA say P = ($Q, \Sigma, \Gamma, S, q_0, Z_0, F$) where the components like Q, Σ etc., are expressed in terms of the known entities V, T, R, S or some other known entities.
(ii) To show that if string $\alpha \in L(G)$, then α is accepted by the PDA constructed by (I) above.
(iii) Conversely, if α is a string accepted by the PDA P constructed above, then $\alpha \in L(G)$

Part I: For the construction of the PDA P = ($Q, \Sigma, \Gamma, S, q_0, Z_0, F$), we should define the values of the various components Q, Σ, etc., in terms of already known components V, T, R, S of the grammar G or some other known or newly introduced symbols.

We define the components of P as follows:

(i) Q = the set of states of PDA = { q }, q is the only state of Q, and q is some *new symbol* not involved in V, T, R and S
(ii) Σ = the set of tape symbols of P, the proposed PDA
= T (the terminals of the *given* grammar G)
(iii) Γ = the stack symbols of P, = (T U V)
= the set of all symbols which are terminal or non-terminals in the given grammar G
(iv) q_0= initial state = q (the only state in Q is naturally the initial state also)
(v) Z_0 = S, the start symbol of the given grammar G
(vi) F = {q},
q being the only state in the PDA P, is naturally the only final state of the PDA.
(vii) Next, we show below how the required function
δ: Q x Σ x $\Gamma \to$ Power Set of (Q x Γ)
is obtained in terms of the known entities Q, V, T, R and S.

(a) For each rule $A \to \beta$ in R of the grammar G with $A \in V$ and $\beta \in (V \cup T)^*$,
we define
$\delta(q, \in, A) = \{ (q, \beta) \mid A \to \beta \text{ is a rule in R} \}$

(Note: There may be more than one rules with the some L.H.S, for example
$A \rightarrow \beta a$ and $A \rightarrow b B C D$)

(b) each (terminal) $a \in T$,
$\delta (q, a, a) = \{(q, \in)\}$

This completes the definition of P, the required PDA. Next our job is to show that L(G), the language generated by G is same as N(P), the language accepted by P, the PDA which we have designed above.

***Proof of Parts II and III* are based on the proof of the following:**

Lemma ***Let*** $S = \Upsilon_0 \rightarrow \Upsilon_1 \ldots\ldots \rightarrow \Upsilon_n = w = a_1 a_2 \ldots\ldots a_n \in L(G)$
be a left-most derivation of w from grammar G,
where

$$\Upsilon_i \rightarrow \Upsilon_{i+1}$$

is obtained by single application of left-most derivation, using some rule of R of the grammar G,
Then, to each Υ_i, there is a unique configuration / ID of the PDA as explained below so that Υ_n corresponds to the configuration of PDA which accepts w:
Let
$\Upsilon_i = x_i \alpha_i$
where $x_i \in T^*$ and $\alpha_i \in (VUT)^*$.
Then
the string Υ_i of the derivation
$\Upsilon_0 = S \Rightarrow \Upsilon_1 \Rightarrow \Upsilon_2 \ldots \Rightarrow \Upsilon_i \ldots \Rightarrow \Upsilon_n = w$
is made to correspond to the ID (y_i, α_i) of the pushdown automata constructed in Part I. The correspondence is diagrammatically shown in figure 17 where y_i is the yet-to-be scanned part of the string w on the tape,
Tape:

Fig.17

and the first terminal in Υ_I is being

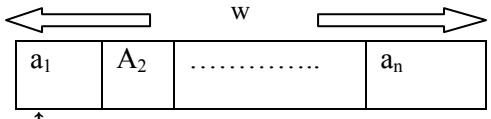
scanned by the Head

Proof of the Lemma

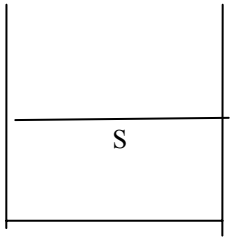
We prove the lemma by induction on i of Υ_i

Base Case: i = 0

$\Upsilon_0 = S$
and initially the Head scans the left-most symbol on the tape, i.e.,



↑
q



thus
$\Upsilon_0 = x_0$ S
where $x_0 = \epsilon$ = empty string
$S \in \Gamma^*$

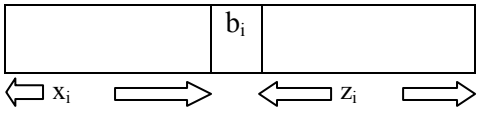
Induction hypothesis:

We assume that if $\Upsilon_j = x_j \alpha_j$ for j = 1, 2, …….i. (where each α_i starts with a non-terminal) for each of $\Upsilon_0, \Upsilon_1, \ldots\ldots, \Upsilon_i$, the correspondence between j^{th}strings Υ_j in the derivation of w from S
and the configuration config (j) given in figure 18

Fig. 18

Induction step

To show that the correspondence is preserved for j = i + 1 also. There is no loss of generality if we assume that if $\alpha_j \neq \epsilon$ then
$\alpha_j = D_j \beta_j$ for j = 1, …….i
where D_j is a non-terminal symbol in the grammar.
Let b_i be the first symbol of y_i (where b_i is one of the a_j's)
i.e $y_i = b_i z_i$
where z_i is a string of terminals.
$\Upsilon_{i+1} = x_i D_{i+1} \xi_{i+1} \alpha_j$
As



$w \in L(G)$

$D_i \rightarrow a_{i1} \ldots\ldots\ldots a_{ik} \quad D_{i+1} \ldots\ldots\ldots$

$\therefore x_i a_{i1} \ldots\ldots a_{ik} D_{i+1} \ldots..$

$x_i a_{i1}$ must be a prefix of at least

then there must be either a production

$D_i \rightarrow b_i \ldots..D_{i+1} \ldots\ldots..$

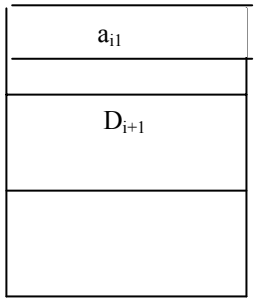
a production in G

or A sequence of production

$D_i \rightarrow D_{i1} \ldots\ldots\ldots\ldots$

$D_{i1} \rightarrow D_{i2} \ldots\ldots\ldots$

$D_{ik} \rightarrow b_i \ldots\ldots$



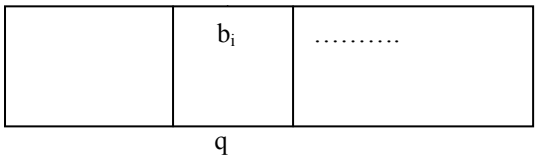
Without loss of generality, we assume that

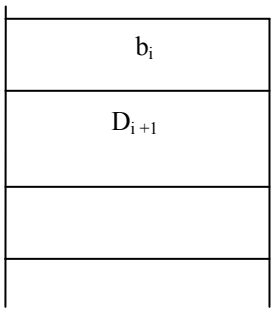
$D_i \rightarrow b_i \ldots\ldots D_{i+1} \ldots\ldots\ldots$, with D_{i+1} being the first non-terminal from left in the production used in Υ_{i+1} from Υ_I

But corresponding to this production there is a move

$S(q, \epsilon, Di) = (q, b_i \ldots\ldots\ldots D_{i+1} \ldots\ldots\ldots)$

using this move the config becomes

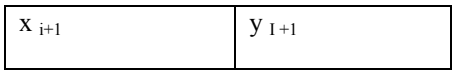


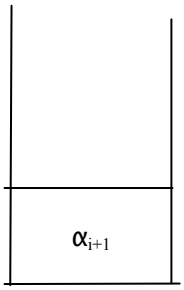


Then all the b's are popped off from the stack and Head of the tape moves to the right of the symbol next to be on the tape by the moves of the type

$\delta(q, b_i, b) = (q, \epsilon)$

Finally $D_{i+1}$is the top configration after the execution of the above moves gives is of the form



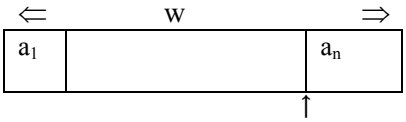


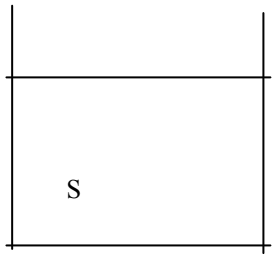
Where

α_{i+1} has a non-terminal as its left most symbol.

This completes the proof of the lemma.

Next, the lemma establishes a one-to-one correspondence between the strings Υ_i in the derivations of w in the grammar G and the configurations of the pushdown automata constructed in Part I, in such a manner that $\Upsilon_n = w$ correspond to the following configuration that indicates acceptance of w and vice-versa.





Stack

This completes the proof of the Part II and Part III.

3.7 SUMMARY

In this unit we have considered the recognition problem and found out whether we can solve it for a larger class of languages. The corresponding accepter for the context-free languages are PDA's. There are some languages which are not context free. We can prove the non-context free languages by using the pumping lemma. Also in this unit we discussed about the equivalence two approaches, of getting a context free language. One approach is using context free grammar and other is Pushdown Automata.

3.8 SOLUTIONS/ANSWERS

Ex.1) (a) $S \rightarrow S[S] \rightarrow [S] \rightarrow [\ ]$

(b) $S \rightarrow S[S] \rightarrow [S] \rightarrow [S[S\ ]] \rightarrow [[S] \rightarrow [[]]$.

Similarly rest part can be done.

Ex.2) (a) $S \rightarrow aSb/aAb$

$A \rightarrow bA/b$

Ex.3) (a) $S \rightarrow aSa/bSb/\wedge$

(b) $S \rightarrow aSa/bSb/a/b$.

Ex.4) (a) $S \rightarrow ASB$ (type 2 production)
$S \rightarrow b$ (type 3 production)
$A \rightarrow aA$ (type 3 production)

So the grammar is of type 2.

(b) $S \rightarrow aSa$ (type 2 production)
$S \rightarrow bSb$ (type 2 production)
$S \rightarrow a$ (type 3 production)
$S \rightarrow b$ (type 3 production)
$S \rightarrow \wedge$ (type 3 production)

So the grammar is of type 2.

(c) Type 2.

Ex.5) (a) $S \rightarrow AB$
$S \rightarrow aAb^5/\wedge$

$B \rightarrow b^7Ba/\wedge$

(b) $S \rightarrow AB$
$A \rightarrow aAb^p/\wedge$
$B \rightarrow b^qBa/\wedge$

Ex.6) Suppose language is $\{wcw^T : w \in \{a,b\}^*\}$ then pda is

(0, a, x, push (a), 0), (0, b, x, push (b), 0),
(0, a, a, push (a), 0), (0, b, a, push (b), 0),
(0, a, b, push (a), 0), (0, b, b, push (b), 0),
(0, c, a, nop, 1), (0, c, b, nop, 1),
(0, c, x, nop, 1), (1, a, a, pop, 1),
(1, b, b, pop, 1), (1, $\wedge$, x, nop, 2),

Ex.7) Language is $\{ww^T : w \in \{a,b\}^*\}$. Similarly as Ex 6.

Ex.8) Apply pumping lemma to get a contradiction. The proof is similar to the proof that the given language is not regular.

Ex.9) Apply pumping lemma to get a contradiction. The proof is similar to the proof that the given language is not regular.