BIJU PATNAIK UNIVERSITY OF TECHNOLOGY, ODISHA

Lecture Notes

On

Markov Chain

Part 1

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Markov Chain Part 1

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Outline

- Stochastic Process
- Markov Chain

Stochastic Process

- A stochastic process is a indexed collection of random variables $\{X_t\} = \{ X_0, X_1, X_2, \ldots \}$ for describing the behavior of a system operating over some period of time.

- For example :
  - $X_0 = 3$, $X_1 = 2$, $X_2 = 1$, $X_3 = 0$, $X_4 = 3$, $X_5 = 1$

Stochastic Process (cont.)

- An inventory example:
- A camera store stocks a particular model camera.
- D_t represents the demand for this camera during week t.
- D_t has a Poisson distribution with a mean of 1.
- X_t represents the number of cameras on hand at the end of week t. ($X_0 = 3$)
- If there are no cameras in stock on Saturday night, the store orders three cameras.
- $\{ X_t \}$ is a stochastic process.
- $X_{t+1} = \max\{ 3 - D_{t+1}, 0 \}$ if $X_t = 0$

  $\max\{ X_t - D_{t+1}, 0 \}$ if $X_t \geq 0$

Markov Chain

- A stochastic process $\{X_t\}$ is a Markov chain if it has Markovian property.

- Markovian property:
  - $P\{ X_{t+1} = j \mid X_0 = k_0, X_1 = k_1, \ldots, X_{t-1} = k_{t-1}, X_t = i \}$
    $= P\{ X_{t+1} = j \mid X_t = i \}$

- $P\{ X_{t+1} = j \mid X_t = i \}$ is called the transition probability.

Markov Chain (con.)

- Stationary transition probability:
  - If ,for each i and j, $P\{ X_{t+1} = j \mid X_t = i \} = P\{ X_1 = j \mid X_0 = i \}$, for all t, then the transition probability are said to be stationary.

Markov Chain (con.)

- Formulating the inventory example:
  - Transition matrix:

$$\mathbf{P} = \begin{array}{c|cccc} \text{state} & 0 & 1 & 2 & 3 \\ \hline 0 & p_{00} & p_{01} & p_{02} & p_{03} \\ 1 & p_{10} & p_{11} & p_{12} & p_{13} \\ 2 & p_{20} & p_{21} & p_{22} & p_{23} \\ 3 & p_{30} & p_{31} & p_{32} & p_{33} \end{array}$$

Markov Chain (con.)

- $X_{t+1} = \max\{ 3 - D_{t+1}, 0 \}$ if $X_t = 0$
  $\max\{ X_t - D_{t+1}, 0 \}$ if $X_t \geq 1$

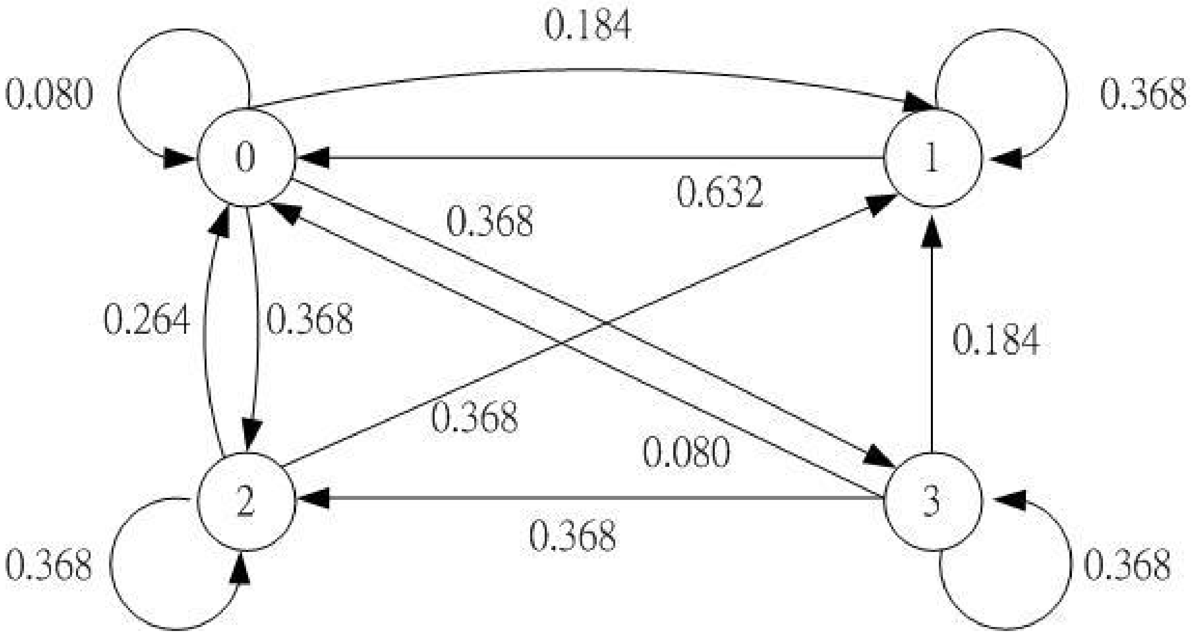
- $p_{03} = P\{ D_{t+1} = 0 \} = 0.368$
- $p_{02} = P\{ D_{t+1} = 1 \} = 0.368$
- $p_{01} = P\{ D_{t+1} = 2 \} = 0.184$
- $p_{00} = P\{ D_{t+1} \geq 3 \} = 0.080$

P =

state	0	1	2	3
0	0.080	0.184	0.368	0.368
1	0.632	0.368	0.000	0.000
2	0.264	0.368	0.368	0.000
3	0.080	0.184	0.368	0.368

Markov Chain (con.)

- The state transition diagram:



Markov Chain (con.)

- n-step transition probability :
  - $p_{ij}^{(n)} = P\{ X_{t+n} = j \mid X_t = i \}$

- n-step transition matrix :

$$
\mathbf{P}^{(n)} = \begin{array}{c|cccc}
\text{state} & 0 & 1 & \cdots & M \\
0 & P_{00}^{(n)} & P_{01}^{(n)} & \cdots & P_{0M}^{(n)} \\
1 & P_{10}^{(n)} & P_{11}^{(n)} & \cdots & P_{1M}^{(n)} \\
\vdots & \cdots & \cdots & \cdots & \cdots \\
M & P_{M0}^{(n)} & P_{M1}^{(n)} & \cdots & P_{MM}^{(n)}
\end{array}
$$

Markov Chain (con.)

- Chapman-Kolmogorove Equation :

$$p_{ij}^{(n)} = \sum_{k=0}^{M} p_{ik}^{(m)} p_{kj}^{(n-m)}$$

for all i = 0, 1, …, M,
j = 0, 1, …, M,
and any m = 1, 2, …, n-1,
n = m+1, m+2, …

- The special cases of m = 1 leads to :

$$p_{ij}^{(n)} = \sum_{k=0}^{M} p_{ik}^{(1)} p_{kj}^{(n-1)}$$

for all i and j

- Thus the n-step transition probability can be obtained from one-step transition probability recursively.

Markov Chain (con.)

- Conclusion :
  - $\mathbf{P}^{(n)} = \mathbf{P}\mathbf{P}^{(n-1)} = \mathbf{P}\mathbf{P}\mathbf{P}^{(n-2)} = \ldots = \mathbf{P}^{n}$

- n-step transition matrix for the inventory example :

P =

state	0	1	2	3
0	0.080	0.184	0.368	0.368
1	0.632	0.368	0.000	0.000
2	0.264	0.368	0.368	0.000
3	0.080	0.184	0.368	0.368

$\mathbf{P}^{(4)}$ =

state	0	1	2	3
0	0.289	0.286	0.261	0.164
1	0.282	0.285	0.268	0.166
2	0.284	0.283	0.263	0.171
3	0.289	0.286	0.261	0.164

Markov Chain (con.)

- What is the probability that the camera store will have three cameras on hand 4 weeks after the inventory system began ?

- $P\{ X_n = j \} = P\{ X_0 = 0 \} p_{0j}^{(n)} + P\{ X_0 = 1 \} p_{1j}^{(n)} + \dots$

  $+ P\{ X_0 = M \} p_{Mj}^{(n)}$

- $P\{ X_4 = 3 \} = P\{ X_0 = 0 \} p_{03}^{(4)} + P\{ X_0 = 1 \} p_{13}^{(4)}$

  $+ P\{ X_0 = 2 \} p_{23}^{(4)} + P\{ X_0 = 3 \} p_{33}^{(4)}$

  $= (1)\, p_{33}^{(4)} = 0.164$

Markov Chain (con.)

- Long-Run Properties of Markov Chain
  - Steady-State Probability

$\mathbf{P} =$

state	0	1	2	3
0	0.080	0.184	0.368	0.368
1	0.632	0.368	0.000	0.000
2	0.264	0.368	0.368	0.000
3	0.080	0.184	0.368	0.368

$\mathbf{P}^{(8)} =$

state	0	1	2	3
0	0.286	0.285	0.264	0.166
1	0.286	0.285	0.264	0.166
2	0.286	0.285	0.264	0.166
3	0.286	0.285	0.264	0.166

Markov Chain (con.)

- The steady-state probability implies that there is a limiting probability that the system will be in each state j after a large number of transitions, and that this probability is independent of the initial state.

- Not all Markov chains have this property.

state	0	1	2	3
0	π_0	π_1	π_2	π_3
1	π_0	π_1	π_2	π_3
2	π_0	π_1	π_2	π_3
3	π_0	π_1	π_2	π_3

Markov Chain (con.)

- Steady-State Equations :

$$\pi_j = \sum_{i=0}^{M} \pi_i p_{ij} \quad \text{for i = 0, 1, …, M}$$

$$\sum_{j=0}^{M} \pi_j = 1$$

- , which consists of M+2 equations in M+1 unknowns.

Markov Chain (con.)

- The inventory example :
- $\pi_0 = \pi_0 p_{00} + \pi_1 p_{10} + \pi_2 p_{20} + \pi_3 p_{30}$,
- $\pi_1 = \pi_0 p_{01} + \pi_1 p_{11} + \pi_2 p_{21} + \pi_3 p_{31}$,
- $\pi_2 = \pi_0 p_{02} + \pi_1 p_{12} + \pi_2 p_{22} + \pi_3 p_{32}$,
- $\pi_3 = \pi_0 p_{03} + \pi_1 p_{13} + \pi_2 p_{23} + \pi_3 p_{33}$,
- $1 = \pi_0 + \pi_1 + \pi_2 + \pi_3$.

- $\pi_0 = 0.080\pi_0 + o.632\pi_1 + 0.264\pi_2 + 0.080\pi_3$,
- $\pi_1 = 0.184\pi_0 + 0.368\pi_1 + 0.368\pi_2 + 0.184\pi_3$,
- $\pi_2 = 0.368\pi_0 + \quad + 0.368\pi_2 + 0.368\pi_3$,
- $\pi_3 = 0.368\pi_0 + \quad + \quad + 0.368\pi_3$,
- $1 = \pi_0 + \pi_1 + \pi_2 + \pi_3$.

- $\pi_0 = 0.286$, $\pi_1 = 0.285$, $\pi_2 = 0.263$, $\pi_3 = 0.166$

Reference

- *Hillier and Lieberman*, “Introduction to Operations Research”, seventh edition, McGraw Hill

- THANK YOU

Q & A