BIJU PATNAIK UNIVERSITY OF TECHNOLOGY, ODISHA

# Lecture Notes

On

# Markov Chain

# Part 2

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# Markov Chain Part 2

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# Outline

- Review
- Classification of States of a Markov Chain
- First passage times
- Absorbing States

# Review

- Stochastic process :
  - A stochastic process is a indexed collection of random variables $\{X_t\} = \{ X_0, X_1, X_2, \dots \}$ for describing the behavior of a system operating over some period of time.

- Markov chain :
  - A stochastic process having the Markovian property,
  - $P\{ X_{t+1} = j \mid X_0 = k_0, X_1 = k_1, \dots, X_{t-1} = k_{t-1}, X_t = i \}$
    $= P\{ X_{t+1} = j \mid X_t = i \}$

- One-step transition probability :
  - $p_{ij} = P\{ X_{t+1} = j \mid X_t = i \}$

# Review (cont.)

- N-step transition probability :
  - $p_{ij}^{(n)} = P\{ X_{t+n} = j \mid X_t = i \}$

- Chapman-Kolmogrove equations :

$$p_{ij}^{(n)} = \sum_{k=0}^{M} p_{ik}^{(m)} p_{kj}^{(n-m)}$$

for all i = 0, 1, …, M,
j = 0, 1, …, M,
and any m = 1, 2, …, n-1,
n = m+1, m+2, …

# Review (cont.)

- One-step transition matrix :

$$
\mathbf{P} = \begin{array}{ccccc}
\text{state} & 0 & 1 & \ldots & M \\
0 & P_{00} & P_{01} & \cdots & P_{0M} \\
1 & P_{10} & P_{11} & \cdots & P_{1M} \\
\vdots & \cdots & \cdots & \cdots & \cdots \\
M & P_{M0} & P_{M1} & \cdots & P_{MM}
\end{array}
$$

# Review (cont.)

- N-step transition probability :

$$
\mathbf{P}^{(n)} = \begin{array}{c|cccc}
\text{state} & 0 & 1 & \dots & M \\
0 & P_{00}^{(n)} & P_{01}^{(n)} & \dots & P_{0M}^{(n)} \\
1 & P_{10}^{(n)} & P_{11}^{(n)} & \dots & P_{1M}^{(n)} \\
\vdots & \dots & \dots & \dots & \dots \\
M & P_{M0}^{(n)} & P_{M1}^{(n)} & \dots & P_{MM}^{(n)}
\end{array}
$$

# Review (cont.)

- Steady-state probability :
  - The steady-state probability implies that there is a limiting probability that the system will be in each state j after a large number of transitions, and that this probability is independent of the initial state.

# Classification of States of a Markov Chain

- Accessible :
  - State j is accessible from state i if $P_{ij}^{(n)} > 0$ for some n ≥ 0.
- Communicate :
  - If state j is accessible from state i and state i is accessible from state j, then states i and j are said to communicate.
  - If state i communicates with state j and state j communicates with state k, then state j communicates with state k.
- Class :
  - The state may be partitioned into one or more separate classes such that those states that communicate with each other are in the same class.

# Classification of States of a Markov Chain (cont.)

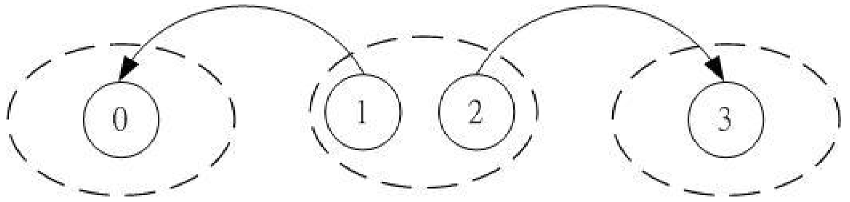
- Irreducible :
  - A Markov chain is said to be irreducible if there is only one class, i.e., all the states communicate.

# Classification of States of a Markov Chain (cont.)

- A gambling example :
  - Suppose that a player has \$1 and with each play of the game wins \$1 with probability p > 0 or loses \$1 with probability 1-p. The game ends when the player either accumulates \$3 or goes broke.

**P =**

| state | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 1 | 0 | 0 | 0 |
| 1 | 1-p | 0 | p | 0 |
| 2 | 0 | 1-p | 0 | p |
| 3 | 0 | 0 | 0 | 1 |



# Classification of States of a Markov Chain (cont.)

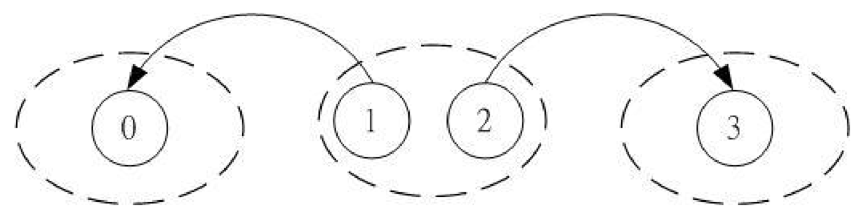
- Transient state :
  - A state is said to be a transient state if, upon entering this state, the process may never return to this state. Therefore, state I is transient if and only if there exists a state j ($j \neq i$) that is accessible from state i but not vice versa.

- Recurrent state :
  - A state is said to be a recurrent state if, upon entering this state, the process definitely will return to this state again. Therefore, a state is recurrent if and only if it is not transient.

# Classification of States of a Markov Chain (cont.)

- Absorbing state :
  - A state is said to be an absorbing state if, upon entering this state, the process never will leave this state again. Therefore, state i is an absorbing state if and only if $P_{ii} = 1$.

$$\mathbf{P} = \begin{array}{c|cccc} \text{state} & 0 & 1 & 2 & 3 \\ \hline 0 & 1 & 0 & 0 & 0 \\ 1 & 1-p & 0 & p & 0 \\ 2 & 0 & 1-p & 0 & p \\ 3 & 0 & 0 & 0 & 1 \end{array}$$



# Classification of States of a Markov Chain (cont.)

- Period :
  - The period of state i is defined to be the integer t (t>1) such that $P_{ii}^{(n)} = 0$ for all value of n other than t, 2t, 3t, … .
  - $P_{11}^{(k+1)} = 0$, k = 0, 1 ,2 , …

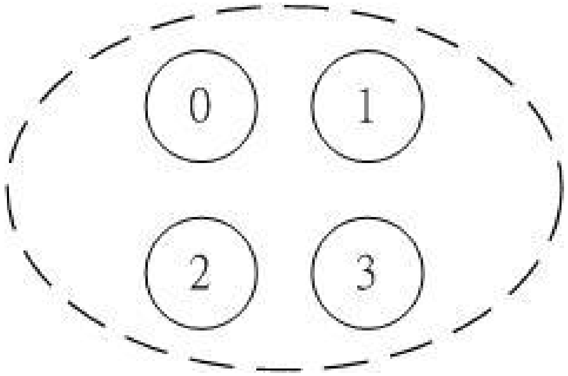
- Aperiodic :
  - If there are two consecutive numbers s and s+1 such that the process can be in the state i at times s and s+1, the state is said to be have period 1 and is called an aperiodic state.

- Ergodic :
  - Recurrent states that are aperiodic are called ergodic states.
  - A Markov chain is said to be ergodic if all its states are ergodic.
  - For any irreducible ergodic Markov chain, steady-state probability, $\lim_{n\to\infty} p_{ij}^{(n)}$ ,exists.

# Classification of States of a Markov Chain (cont.)

- An inventory example :
  - The process is irreducible and ergodic and therefore, has steady-state probability.

$$\mathbf{P} = \begin{array}{c|cccc} \text{state} & 0 & 1 & 2 & 3 \\ \hline 0 & 0.080 & 0.184 & 0.368 & 0.368 \\ 1 & 0.632 & 0.368 & 0.000 & 0.000 \\ 2 & 0.264 & 0.368 & 0.368 & 0.000 \\ 3 & 0.080 & 0.184 & 0.368 & 0.368 \end{array}$$



# First Passage Times

- First Passage time :
  - The first passage time from state i to state j is the number of transitions made by the process in going from state i to state j for the first time.
- Recurrence time :
  - When j = i, the first passage time is just the number of transitions until the process returns to the initial state i and called the recurrence time for state i.
- Example :
  - $X_0 = 3$, $X_1 = 2$, $X_2 = 1$, $X_3 = 0$, $X_4 = 3$, $X_5 = 1$
  - The first passage time from state 3 to state 1 is 2 weeks.
  - The recurrence time for state 3 is 4 weeks.

# First Passage Times (cont.)

- $f_{ij}^{(n)}$ :
  - denotes the probability that the first passage time from state i to state j is n.

- Recursive relationship :

$$f_{ij}^{(n)} = \sum_{k \neq j} p_{ik} f_{kj}^{(n-1)}$$

$$f_{ij}^{(1)} = p_{ij}^{(1)} = p_{ij}$$

$$f_{ij}^{(2)} = \sum_{k \neq j} p_{ik} f_{kj}^{(1)}$$

# First Passage Times (cont.)

- The inventory example :
  - $f_{30}^{(1)} = p_{30} = 0.080$
  - $f_{30}^{(2)} = p_{31} f_{10}^{(1)} + p_{32} f_{20}^{(1)} + p_{33} f_{30}^{(1)}$
    $= 0.184(0.632) + 0.368(0.264) + 0.368(0.080) = 0.243$
  - … …

- Sum :

$$\sum_{n=1}^{\infty} f_{ij}^{(n)} \leqq 1$$

# First Passage Times (cont.)

- Expected first passage time :
- $\mu_{ij}$ = $\infty$ if $\sum_{n=1}^{\infty} f_{ij}^{(n)} < 1$

  $\sum_{n=1}^{\infty} n f_{ij}^{(n)}$ If $\sum_{n=1}^{\infty} f_{ij}^{(n)} = 1$

- $\mu_{ij} = 1 + \sum_{k \neq j} p_{ik} \mu_{kj}$

# First Passage Times (cont.)

- The inventory example :
  - $\mu_{30} = 1 + p_{31}\mu_{10} + p_{32}\mu_{20} + p_{33}\mu_{30}$
  - $\mu_{20} = 1 + p_{21}\mu_{10} + p_{22}\mu_{20} + p_{23}\mu_{30}$
  - $\mu_{10} = 1 + p_{11}\mu_{10} + p_{12}\mu_{20} + p_{13}\mu_{30}$

  - $\mu_{10}$ = 1.58 weeks, $\mu_{20}$ = 2.51 weeks, $\mu_{30}$ = 3.50 weeks

# Absorbing states

- Absorbing states :
  - A state k is called an absorbing state if $p_{kk} = 1$, so that once the chain visits k it remains there forever.

- An gambling example :
  - Suppose that two players (A and B), each having \$2, agree to keep playing the game and betting \$1 at a time until one player is broke. The probability of A winning a single bet is 1/3.

# Absorbing states (cont.)

- The transition matrix form A's point of view

$$
P = \begin{array}{c|ccccc}
\text{state} & 0 & 1 & 2 & 3 & 4 \\
\hline
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 2/3 & 0 & 1/3 & 0 & 0 \\
2 & 0 & 2/3 & 0 & 1/3 & 0 \\
3 & 0 & 0 & 2/3 & 0 & 1/3 \\
4 & 0 & 0 & 0 & 0 & 1
\end{array}
$$

# Absorbing states (cont.)

- Probability of absorption :
  - If k is an absorbing state, and the process starts in state i, the probability of ever going to state k is called the probability of absorption into state k, given the system started in state i.

$$f_{ik} = \sum_{j=0}^{M} p_{ij} f_{jk} \qquad \text{for } i = 0, 1, 2, \ldots, M$$

subject to the conditions

$f_{kk} = 1$,

$f_{ik} = 0$, if state i is the recurrent and $i \neq k$.

- The gambling example :
  - $f_{20} = 4/5$, $f_{24} = 1/5$

# Reference


- *Hillier and Lieberman*, “Introduction to Operations Research”, seventh edition, McGraw Hill

- THANK YOU